System of Linear Equations

Slide for MA1203 Business Mathematics II

Week 1 & 2

Function

A manufacturer would like to know how his company's profit is related to its production level.

How does one quantity depend on another? The relationship between two quantities is conveniently described in mathematics by using the concept of a function.

A **function** f is a rule that assigns to each value of x one and only one value of y.

Given the function $y = f(x)$. The variable x is referred to as the **independent variable**, and the variable y is called the **dependent variable.** The set of all values that may be assumed by x is called the **domain** of the function f, and the set comprising all the values assumed by $y = f(x)$ as x takes on all possible values in its domain is called the **range** of the function f.

Linear Function

The function f defined by

$$y = f(x) = ax + b$$

where a and b are constants, is called a **linear function.**

The graph of the function is a straight line in the plane.

Linear functions play an important role in the quantitative analysis of business and economic problems.

Linear Cost, Revenue, and Profit Functions

Cost, Revenue, and Profit Functions

Let x denote the number of units of a product manufactured or sold. Then, the **total cost function** is

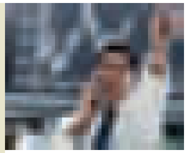
$$C(x) = \text{Total cost of manufacturing } x \text{ units of the product}$$

The **revenue function** is

$$R(x) = \text{Total revenue realized from the sale of } x \text{ units of the product}$$

The **profit function** is

$$P(x) = \text{Total profit realized from manufacturing and selling } x \text{ units of the product}$$

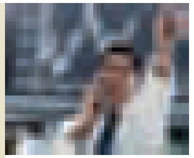


APPLIED EXAMPLE 3 Profit Functions Puritron, a manufacturer of water filters, has a monthly fixed cost of \$20,000, a production cost of \$20 per unit, and a selling price of \$30 per unit. Find the cost function, the revenue function, and the profit function for Puritron.

Linear Demand Function

A **demand equation** expresses relationship between the unit price and the quantity demanded. The corresponding graph of the demand equation is called a **demand curve.** In general, the quantity demanded of a commodity decreases as its unit price increases, and vice versa.

A **demand function** is defined by $p = f(x)$, where p measures the unit price and x measures the number of units of the commodity. It is generally characterized as a decreasing function of x; that is, $p = f(x)$ decreases as x increases.

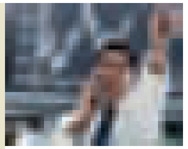


APPLIED EXAMPLE 4 Demand Functions The quantity demanded of the Sentinel iPod™ alarm clock is 48,000 units when the unit price is \$8. At \$12 per unit, the quantity demanded drops to 32,000 units. Find the demand equation, assuming that it is linear. What is the unit price corresponding to a quantity demanded of 40,000 units? What is the quantity demanded if the unit price is \$14?

Linear Supply Function

An equation that expresses the relationship between the unit price and the quantity supplied is called a **supply equation,** and the corresponding graph is called a **supply curve.**

A **supply function,** defined by $p = f(x)$, is generally characterized by an increasing function of x; that is, $p = f(x)$ increases as x increases.



APPLIED EXAMPLE 5 Supply Functions The supply equation for a commodity is given by $4p - 5x = 120$, where p is measured in dollars and x is measured in units of 100.

a. Sketch the corresponding curve.
b. How many units will be marketed when the unit price is $55?

System of Linear Equations

In many economics problems, there are two or more linear equations that must be satisfied *simultaneously*. Then we have a **system of linear equations** in two or more variables.

Consider a system of two linear equations in two variables.

$$ax + by = h$$

$$cx + dy = k$$

where a, b, c, d, h, and k are real constants and neither a and b nor c and d are both zero.

7.1 Solving System of Linear Equations

Graphing Solutions

The solution of a system of two linear equations could be found by finding the point of intersection of two straight lines.

Suppose we are given two straight lines L_1 and L_2 with equations

$$y = m_1x + b_1 \text{ and } y = m_2x + b_2$$

(where m_1, b_1, m_2, and b_2 are constants) that intersect at the point $P(x_0,y_0)$.
The point $P(x_0,y_0)$ lies on the line L_1, so it satisfies the equation $y = m_1x + b_1$. It also lies on the line L_2, so it satisfies the equation $y = m_2x + b_2$.

EXAMPLE

Find a solution to the two linear equations

$$x + y = 10 \text{ and } x - y = 0.$$

Solutions by Substitution and Elimination

EXAMPLE

Solve the following system of equations

$$5x + 2y = 3 \text{ and } -x - 4y = 3$$

(a) by substitution and

(b) by elimination.

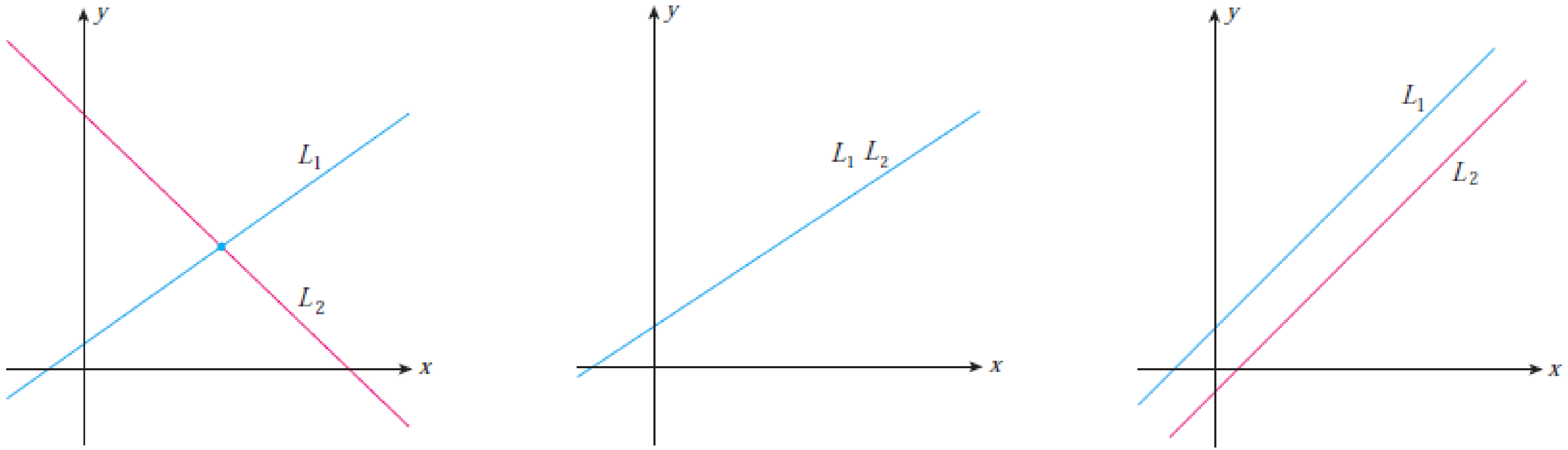
Solutions of System of Linear Equations

Given two lines L_1 and L_2, *one and only one* of the following may occur:

a. L_1 and L_2 intersect at exactly one point.

b. L_1 and L_2 are parallel and coincident.

c. L_1 and L_2 are parallel and distinct.



(a) Unique solution (b) Infinitely many solutions (c) No solution

Examples

1. A system of equations with exactly one solution Consider the system

$$\begin{aligned} 2x - y &= 1 \\ 3x + 2y &= 12 \end{aligned}$$

2. A system of equations with infinitely many solutions Consider the system

$$\begin{aligned} 2x - y &= 1 \\ 6x - 3y &= 3 \end{aligned}$$

3. A system of equations that has no solution Consider the system

$$\begin{aligned} 2x - y &= 1 \\ 6x - 3y &= 12 \end{aligned}$$

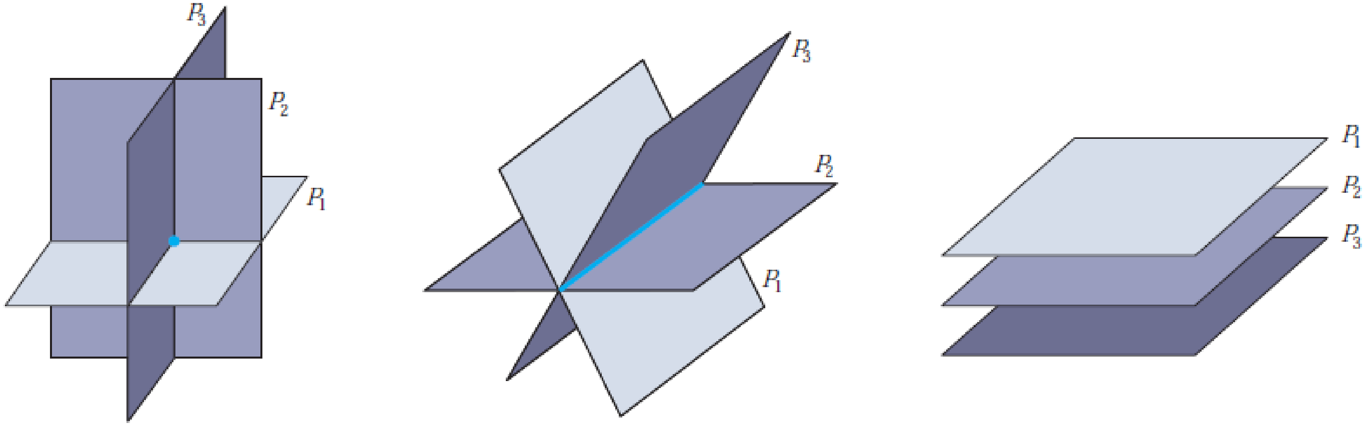
Solutions of Systems of Equations

A linear system composed of three linear equations in three variables *x*, *y*, and *z* has the general form

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad (2)$$

Each equation in System (2) represents a *plane* in three-dimensional space, and the *solution(s) of the system* is precisely the point(s) of intersection of the three planes defined by the three linear equations that make up the system.

The system has one and only one solution, infinitely many solutions, or no solution, depending on whether and how the planes intersect one another.



(a) A unique solution (b) Infinitely many solutions (c) No solution

7.2 Linear Equations in n Variables

Linear Equations in *n* Variables

Linear Equations in *n* Variables

A linear equation in n variables, $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$$

where $a_1, a_2, \ldots, a_n$ (not all zero) and c are constants.

Procedures to solve the equations are *subtracting equations* and *multiplying equations by a constant*.

Solution by Row Operations

The idea is to transform a given system of equations into another with the same mathematical properties and hence the same solution.

The aim is to transform a system in such a way as to produce a *simpler system* which is easier to solve.

Three types of operations are permitted:

1. Multiply an equation by a nonzero constant.
2. Add a multiple of one equation to another.
3. Interchange two equations.

EXAMPLE 1 Solve the following system of equations:

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

Augmented Matrices

With the aid of **matrices,** which are rectangular arrays of numbers, we can eliminate writing the variables at each step of the reduction and thus save ourselves a great deal of work.

For example, the system

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 3x + 8y + 5z &= 27 \\ -x + y + 2z &= 2 \end{aligned}$$

may be represented by the matrix

$$\left[\begin{array}{rrr|r} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

The submatrix consisting of the first three columns of the Matrix is called the **coefficient matrix** of the System. The matrix itself is referred to as the **augmented matrix** of the System since it is obtained by joining the matrix of coefficients to the column (matrix) of constants. The vertical line separates the column of constants from the matrix of coefficients.

Row-reduced form of a matrix

Row-Reduced Form of a Matrix

1. Each row consisting entirely of zeros lies below all rows having nonzero entries.
2. The first nonzero entry in each (nonzero) row is 1 (called a **leading 1**).
3. In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
4. If a column in the coefficient matrix contains a leading 1, then the other entries in that column are zeros.

EXAMPLE 3 Determine which of the following matrices are in row-reduced form. If a matrix is not in row-reduced form, state the condition that is violated.

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ **c.** $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 0 & 1 & 2 & -2 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$ **e.** $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{array}\right]$ **f.** $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

g. $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \end{array}\right]$

2.2 Systems of Linear Equations: Unique Solutions

The Gauss–Jordan Method

The **Gauss–Jordan elimination method** is systematic procedure which always leads to a matrix in row reduced form. It is a suitable technique for solving systems of linear equations of any size. One advantage of this technique is its adaptability to the computer.

This method involves a sequence of operations on a system of linear equations to obtain at each stage an **equivalent system**—that is, a system having the same solution as the original system. The reduction is complete when the original system has been transformed so that it is in a certain standard form from which the olution can be easily read.

The operations of the Gauss–Jordan elimination method are

1. Interchange any two equations.

2. Replace an equation by a nonzero constant multiple of itself.

3. Replace an equation by the sum of that equation and a constant multiple of any other equation.

Row Operations

We need an adaptation of the Gauss–Jordan elimination method in solving systems of linear equations using matrices.

The three operations on the equations of a system translate into the following

Row Operations

1. Interchange any two rows.
2. Replace any row by a nonzero constant multiple of itself.
3. Replace any row by the sum of that row and a constant multiple of any other row.

Notations for Gauss-Jordan Elimination using Matrices

A column in a coefficient matrix is called a **unit column** if one of the entries in the column is a 1 and the other entries are zeros.

Notation for Row Operations

Letting R_i denote the ith row of a matrix, we write:

Operation 1 $R_i \leftrightarrow R_j$ to mean: Interchange row i with row j.

Operation 2 cR_i to mean: Replace row i with c times row i.

Operation 3 $R_i + aR_j$ to mean: Replace row i with the sum of row i and a times row j.

The Gauss-Jordan Elimination Method

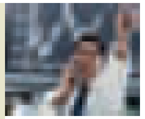
The Gauss–Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows (operation 1), if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

Examples

EXAMPLE 5 Solve the system of linear equations given by

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$



APPLIED EXAMPLE 1 **Manufacturing: Production Scheduling** Ace Novelty wishes to produce three types of souvenirs: Types *A*, *B*, and *C*. To manufacture a Type *A* souvenir requires 2 minutes on Machine I, 1 minute on Machine II, and 2 minutes on Machine III. A Type *B* souvenir requires 1 minute on Machine I, 3 minutes on Machine II, and 1 minute on Machine III. A Type *C* souvenir requires 1 minute on Machine I and 2 minutes each on Machines II and III. There are 3 hours available on Machine I, 5 hours available on Machine II, and 4 hours available on Machine III for processing the order. How many souvenirs of each type should Ace Novelty make in order to use all of the available time?

2.3 Systems of Linear Equations: Underdetermined and Overdetermined Systems

A System of Equations with an Infinite Number of Solutions

EXAMPLE 1 A System of Equations with an Infinite Number of Solutions Solve the system of linear equations given by

$$\begin{aligned} x + 2y &= 4 \\ 3x + 6y &= 12 \end{aligned} \tag{9}$$

EXAMPLE 2 A System of Equations with an Infinite Number of Solutions Solve the system of linear equations given by

$$\begin{aligned} x + 2y - 3z &= -2 \\ 3x - y - 2z &= 1 \\ 2x + 3y - 5z &= -3 \end{aligned} \tag{10}$$

A System of Equations That Has No Solution

EXAMPLE 3 A System of Equations That Has No Solution Solve the system of linear equations given by

$$\begin{aligned} x + y + z &= 1 \\ 3x - y - z &= 4 \\ x + 5y + 5z &= -1 \end{aligned} \tag{11}$$

EXAMPLE 4 A System with More Equations Than Variables Solve the following system of linear equations:

$$\begin{aligned} x + 2y &= 4 \\ x - 2y &= 0 \\ 4x + 3y &= 12 \end{aligned}$$

EXAMPLE 5 A System with More Variables Than Equations Solve the following system of linear equations:

$$\begin{aligned} x + 2y - 3z + w &= -2 \\ 3x - y - 2z - 4w &= 1 \\ 2x + 3y - 5z + w &= -3 \end{aligned}$$