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THE PARTITION DIMENSION FOR A SUBDIVISION OF HOMOGENEOUS CATERPILLARS

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The partition dimension for a subdivision of homogeneous caterpillars

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Abstract

The concept of graph partition dimension was introduced by Chartrand et al. (1998). Let $G = (V, E)$ be a connected graph. For every $v \in V(G)$ and $L \subseteq V(G)$, define the distance from v to L as $d(v, L) = min\{d(v, w)|w \in L\}$. Let $\Pi = \{L_1, L_2, \cdots, L_k\}$ be a partition of $V(G)$. The representation of a vertex v with respect to Π is defined as $r(v|\Pi) = (d(v, L_1), d(v, L_2), \cdots, d(v, L_k))$. The partition Π is called a resolving partition of G if all representations of the vertices are distinct. The partition dimension of a graph G can be defined as the cardinality of a minimum resolving partition Π of G.

Let $e \in V(G)$ and $k \geq 1$. The subdivision $S(G(e; k))$ of a graph G on e is a graph obtained from graph G by replacing edge e with a path on $k + 2$ vertices. In this paper, we determine the partition dimension of $S(G(e; k))$ with $G \simeq C(m; r)$ is a homogeneous caterpillar. We show that $pd(S(G(e; k))) = pd(G)$ for almost all values of m, r and k.

Keywords: Caterpillar, subdivision, homogeneous, partition dimension, resolving partition.

2000 Mathematics Subject Classification: 05C12

1. Introduction

The concept of the metric dimension of a graph was introduced independently by Slater [9] and Harary and Melter [6]. Let $G = (V, E)$ be a connected graph. For $v, w \in V(G)$ and $A \subseteq V(G)$, the *distance* $d(v, w)$ from vertices v and w is the length of a shortest path from v to w. The *distance* $d(v, A)$ from vertex v to A is defined as $\min\{d(v, a)|a \in A\}$. The *representation* $r(v|A)$ of v with respect to A is the vector $(d(v, a_1), d(v, a_2), \cdots, d(v, a_k))$ if $A = \{a_1, a_2, \cdots, a_k\}$. The set A is called a *resolving set* of G if $r(v|A) \neq r(w|A)$ for all distinct $v, w \in V(G)$. The *metric dimension* $\beta(G)$ of G is the cardinality of a minimum resolving set. One of variants of this concept, namely partition dimension, was then introduced by Chartrand et al. [2]. Let $\Pi = \{L_1, L_2, \cdots, L_k\}$ be a *k-partition* of $V(G)$. The *representation* of a vertex v with respect to Π, denoted by $r(v|\Pi)$, is the vector $(d(v, L_1), d(v, L_2), \cdots, d(v, L_k))$. The partition Π is called a *resolving partition* of G if $r(w|\Pi) \neq r(v|\Pi)$ for all distinct vertices in G. Two vertices

u, v in G are said to be *distinguishable* if there exists a partition class L such that $d(u, L) \neq d(v, L)$. In this case, vertices u and v are also called *distinguished* by L. A vertex $v \in G$ is called a *dominant* vertex if the distance from v to any other partition class induced by a resolving partition in G is 1. The *partition dimension* $pd(G)$ of G is the smallest integer k such that G has a resolving k-partition. If two distinct vertex u and v are in the same partition class under Π, then we write $u \sim_{\Pi} v$, otherwise $u \nsim_{\Pi} v$.

Finding the partition dimension of a general graph is a NP-*complete* problem. So, there is no efficient algorithm to determine the partition dimension of a general graph. Therefore, many researchers restrict their study in finding the partition dimension of graphs for certain classes.

Chartrand et al. [3] showed that path P_n with $n \geq 2$ is the only graph G with $pd(G) = 2$. The complete graph K_n is the only graph G with $pd(G) = n$. Furthermore, they show that the graphs $K_{1,n-1}, K_n - e$ and $K_1 + (K_1 \cup K_{n-2})$ are the only graphs G on n vertices with $pd(G) = n - 1$. Tomescu [7] showed that there are 23 graphs having $pd(G) = n - 2$.

The partition dimension of a graph produced by a binary graph operation has been also considered. For instance, Baskoro and Darmaji [1] determined the partition dimension of the corona product of two graphs. For wheels, Tomescu et al. [8] gave the lower and upper bounds of the partition dimension of wheels W_n, namely $\lceil (2n)^{1/3} \rceil \leq pd(W_n) \leq p + 1$, with p be a smallest prime such that $p(p-1) \geq n$. For $3 \leq n \leq 20$, they gave the exact values of their partition dimensions.

There are many open problems in determining the partition dimension of graphs. One of them is finding this value for an arbitrary tree. Until now, there is no explicit formula known for the partition dimension of a general tree. However, for some particular classes of trees for instance paths [3], caterpillars [5], firecrackers and banana trees [4], their values are known. Chartrand et al. [2] showed that the partition dimension of double star $S_{m,n}$ depends on their maximum degree. Moreover, they also derived the lower and upper bounds of a caterpillar, namely a tree having the property that the removal of its leaves results a path. In this paper, we present the partition dimension of a subdivision of a homogeneous caterpillar. Note that the subdivision of a tree is also a tree.

2. Previous Lemmas

This lemma is very useful in determining our main results.

Lemma 2.1. [2] *Let G be a connected with a resolving partition Π. If $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then vertices u, v must be in distinct partition classes of* Π.

The following result is a direct consequence of Lemma 2.1

Corollary 2.2. [2] *If connected graph G has a vertex having k leaves, then $pd(G) \geq k$.*

For integers $m, n_1, n_2, \cdots, n_m \geq 2$, define a *caterpillar* $C(m; n_1, n_2, \cdots, n_m)$ as a graph obtained by attaching n_i vertices to each vertex v_i of the path P_m, for $i \in [1, m]$. The path P_m in $C(m; n_1, n_2, \cdots, n_m)$ is called the *backbone* of the caterpillar. All vertices of degree one are called *leaves*. All leaves attached to v_i are labeled by $w_{i1}, w_{i2}, \cdots, w_{in_i}$. Edges $v_i w_{ij}$ and $v_i v_{i+1}$ are called *pendant* and *backbone edges*, respectively. If $n_1 = n_2 = \cdots = n_m = r$, then the caterpillar is called a *homogeneous* caterpillar, and it is denoted by $C(m; r)$. Darmaji et al. [5] gave the partition dimension of a homogeneous caterpillar in the following theorem.

Theorem 2.3. [5] *Let* $G \approx C(m, r)$ *with* $r \geq 3, m \geq 2$. *Then,*

$$pd(G) = \begin{cases} r & if \ r \leq m, \\ r+1 & otherwise. \end{cases}$$

3. Main Results

In this section, we determine the partition dimension of a subdivision graph of a homogeneous caterpillar $C(m, r)$. Let $G = (V, E)$ be a connected graph and $e \in E(G)$. The *subdivision* of a graph G on the edge e, denoted by $S(G(e; k))$, is a graph obtained from the graph G by replacing edge e with a path on $k+2$ vertices. The internal vertices of the path replacing edge e are called *subdivision vertices* in $S(G(e; k))$. The subdivision vertices of $S(G(e; k))$ are labelled by $x_1, x_2, \cdots, x_k$.

Lemma 3.1. *Let* $G \simeq C(m; r_1, r_2, \cdots, r_m)$, $r = max\{r_1, r_2, \cdots, r_m\}$, *and* $r \leq m$. *Let* $e = v_a v_b$ *be a backbone edge of* G, *for some* $a, b \in [1, m]$. *Let* Π *be a resolving* r*-partition of* $S(G(e; k))$ *with* $k \geq 2$. *Let* $(v_a, x_1) \in E(S(G(e; k)))$.
(i) If vertex v_a *has* r *leaves, then* $x_2 \nsim_\Pi v_a$ *and* $x_1 \nsim_\Pi x_2$.
(ii) If vertex v_b *has* r *leaves, then* $x_{k-1} \nsim_\Pi v_b$ *and* $x_k \nsim_\Pi x_{k-1}$.

Proof. Let Π be a resolving r-partition of $S(G(e; k))$. Since vertex v_a has r leaves $w_{a1}, w_{a2}, w_{a3}, \cdots, w_{ar}$, then by Lemma 2.1, $w_{a1}, w_{a2}, w_{a3}, \cdots, w_{ar}$ must be in distinct partition classes. Furthermore, there are (not necessarily distinct) integers $p, q \in [1, r]$ such that $v_a \sim_\Pi w_{ap}$ and $x_1 \sim_\Pi w_{aq}$.

(i) To show the first case, assume that $x_2 \sim_\Pi v_a$. But, this implies that $x_1 \sim_\Pi w_{ai}$ for some $i \in [1, r]$ and $r(w_{ai}|\Pi) = r(x_1|\Pi)$, a contradiction. Therefore $x_2 \nsim_\Pi v_a$. But, this implies that $x_1 \nsim_\Pi x_2$. Since otherwise, there exists $i \in [1, r]$ such that $r(w_{ai}|\Pi) = r(x_1|\Pi)$, a contradiction.

(ii) The second case can be proved similarly.

□

Lemma 3.2. *Let* $G \simeq C(m; r_1, r_2, \cdots, r_m)$, $r = max\{r_1, r_2, \cdots, r_m\}$. *Let* v_a *and* v_b *be two vertices adjacent to* r *leaves, respectively. Let* Π *be a resolving* r*-partition of* $S(G(e,k))$ *with* e *is a backbone edge incident to* v_a *and* $(v_a, x_1) \in E(S(G(e;k)), k \geq 2$. *If* $x_2 \sim_\Pi v_b$ *and* $v_a \nsim_\Pi v_b$, *then* $v_a \nsim_\Pi x_1$.

Proof. Let Π be a resolving r-partition of $S(G(e;k))$. Since v_b has r leaves, there is a leaf w_{bs} such that $x_1 \sim_\Pi w_{bs}$ for some $s \in [1, r]$. Assume that $v_a \sim_\Pi x_1$. Since $x_2 \sim_\Pi v_b$ and $v_a \nsim_\Pi v_b$, then we have $r(w_{as}|\Pi) = r(x_1|\Pi)$, a contradiction. Thus, $v_a \nsim_\Pi x_1$. □

Lemma 3.3. *Let* $G \simeq C(2, 3)$. *If* e *is a backbone edge of* G *and* $k \geq 1$ *then*

$$pd(S(G(e;k))) = \begin{cases} pd(G) + 1 & \text{if } k = 4, \\ pd(G) & \text{otherwise.} \end{cases}$$

Proof. From [5], we know that $pd(G) = 3$ with $\Pi = \{L_1, L_2, L_3\}$ is a resolving partition of G where $L_1 = \{v_1, w_{11}, w_{21}\}$, $L_2 = \{v_2, w_{12}, w_{22}\}$, and $L_1 = \{w_{13}, w_{23}\}$. Now, consider the following two cases:

Case 1. $k = 4$.

We will show that $pd(S(G(e;4))) \geq pd(G) + 1 = 4$. For a contradiction, let $pd(S(G(e;4))) = 3$ and $\Pi' = \{L'_1, L'_2, L'_3\}$ be a resolving partition of $S(G(e;4))$. Without loss of generality, let vertices $v_1 \in L'_1$ and $v_2 \in L'_2$. By Lemma 3.1, $x_2 \in L'_2$ or $x_2 \in L'_3$. Now, we consider the following two subcases.

Subcase 1.1: $x_2 \in L'_2$.

By Lemmas 3.1 and 3.2, we have $x_1 \in L'_3$. Since $v_2 \in L'_2$, by Lemma 3.1, we obtain $x_3 \in L'_1$ or $x_3 \in L'_3$. Clearly that $x_3 \notin L'_1$ otherwise $r(v_2|\Pi') = r(x_2|\Pi')$, so $x_3 \in L'_3$. By Lemma 3.1, x_4 must be in L'_1 or L'_2. If $x_4 \in L'_1$, then we have $r(x_4|\Pi') = r(v_1|\Pi')$. If $x_4 \in L'_2$, then we have $r(x_4|\Pi') = r(x_2|\Pi')$. This implies that $x_2 \notin L'_2$.

Subcase 1.2: $x_2 \in L'_3$.

By Lemma 3.1, we have $x_1 \in L'_2$ or $x_1 \in L'_1$. If $x_1 \in L'_2$, then $r(x_1|\Pi') = r(v_2|\Pi')$. If $x_1 \in L'_1$, then we consider vertices x_3 and x_4 in partition class of Π'. Since $v_2 \in L'_2$, by Lemma 3.1, $x_3 \notin L'_2$ and $x_3 \nsim_\Pi x_4$, so we have four subcases for x_3 and x_4 namely a) $x_3 \in L'_1$ and $x_4 \in L'_2$, and so $r(x_3|\Pi') = r(v_1|\Pi')$. b) $x_3 \in L'_1$ and $x_4 \in L'_3$, and so $r(x_1|\Pi') = r(x_3|\Pi')$. c) $x_3 \in L'_3$ and $x_4 \in L'_1$, and so $r(x_3|\Pi') = r(v_{13}|\Pi')$. d) $x_3 \in L'_3$ and $x_4 \in L'_2$, and so $r(x_3|\Pi') = r(v_{23}|\Pi')$. These imply that $x_2 \notin L'_3$. By Subcases 1.1 and 1.2, we conclude that $pd(S(G(e;k)) \geq 4$.

Furthermore, let $\Pi' = \{L'_1, L'_2, L'_3, L'_4\}$ be a partition of $S(G(e;k))$ where $L'_i = L_i$ for $i \in \{1, 2, 3\}$, and $L'_4 = \{x_1, x_2, x_3, x_4\}$. If u, v are non subdivision vertices of $S(G(e;k))$, then clearly $r(u|\Pi') \neq r(v|\Pi')$ since $r(u|\Pi) \neq r(v|\Pi)$.

If u, v are subdivision vertices of $S(G(e;k))$, then u and v are distinguished by L'_1 or L'_2, so $r(u|\Pi') \neq r(v|\Pi')$. This implies that Π' is a resolving partition of $S(G(e;k))$.

So, $pd(S(G(e;k))) \leq pd(G)+1 = 4$. As a consequence, we have $pd(S(G(e;k))) = pd(G)+1 = 4$.

Case 2. $k \neq 4$.

We will show that $pd(S(G(e;k))) = pd(G) = 3$. Let $\Pi' = \{L'_1, L'_2, L'_3\}$ be a partition of $V(S(G(e;k)))$.

For $k=1$, define $L'_1 = L_1$, $L'_2 = L_2$ and $L'_3 = L_3 \cup \{x_1\}$. For $k=2$, define $L'_1 = L_1 \cup \{x_1\}$, $L'_2 = L_2$ and $L'_3 = L_3 \cup \{x_2\}$. For $k \geq 3$ and $k \neq 4$, define $L'_1 = L_1 \cup \{x_1\}$, $L'_2 = L_2 \cup \{x_k\}$ and $L'_3 = L_3 \cup \{x_2, x_3, \cdots, x_{k-1}\}$.

For $k = 1,2,3$, it is clear that $r(u|\Pi') \neq r(v|\Pi')$ for any distinct vertices u and v. Now consider $k \geq 5$. If u,v are any two subdivision vertices, then $r(u|\Pi') \neq r(v|\Pi')$ because u,v are distinguished by either L'_1 or L'_2. Now, let u be a subdivision vertex and v be a non subdivision vertex. If $u = x_1$ or $u = x_k$, then u,v are distinguished by either L'_3 or L'_2. If $u = x_i$ for some $i \in [2, k-1]$, then u,v are distinguished by either L'_1 or L'_2. This implies that $r(u|\Pi') \neq r(v|\Pi')$. Therefore, $pd(S(G(A;k))) = pd(G) = 3$ for $k \in [1,3]$ or $k \geq 5$. □

Lemma 3.4. *Let $G \simeq C(3;3)$. If e is a backbone edge of G, then $pd(S(G(e;k))) = pd(G)+1$, for $k \geq 1$.*

Proof. From [5], we know that $pd(G) = 3$. Let $\Pi = \{L_1, L_2, L_3\}$ be a resolving partition of G, where $L_1 = \{v_1, w_{11}, w_{21}, w_{31}\}$, $L_2 = \{v_2, w_{12}, w_{22}, w_{32}\}$ and $L_3 = \{v_3, w_{13}, w_{23}, w_{33}\}$. We will show that $pd(S(G(e;k)) \geq pd(G)+1 = 4$.

For a contradiction, assume that $\Pi' = \{L'_1, L'_2, L'_3\}$ is a resolving partition of $S(G(e;k))$. Let $e = v_1v_2$ and without loss of generality assume $v_1 \in L'_1$, $v_2 \in L'_2$, and $v_3 \in L'_3$ (since they are dominant vertices and must be in different partition classes).

For $k=1$. If $x_1 \in L'_1$, then $r(x_1|\Pi') = r(v_{2i}|\Pi')$ for some i. If $x_1 \in L'_2$, then $r(x_1|\Pi') = r(v_{1i}|\Pi')$ for some i. If $x_1 \in L'_3$, then $r(x_1|\Pi') = r(v_3|\Pi')$, a contradiction. Therefore, $pd(S(G(e;k))) \geq 4$ for $k=1$.

For $k \geq 2$. By Lemma 3.1, x_2 must be in either L'_2 or L'_3. If vertex $x_2 \in L'_2$, then by Lemmas 3.1 and 3.2, we must have $x_1 \in L'_3$; so $r(x_1|\Pi') = r(v_3|\Pi')$. This implies that $x_2 \in L'_3$. However, by Lemmas 3.1 and 3.2, we must have $x_1 \in L'_2$, so $r(x_1|\Pi') = r(v_2|\Pi')$, a contradiction. Therefore, $pd(S(G(e;k))) \geq 4$ for $k \geq 2$.

Now, we will show $pd(S(G(e;k))) \leq pd(G)+1$. To show this, consider the partition $\Pi' = \{L'_1, L'_2, L'_3, L'_4\}$ of $S(G(e;k))$ where $L'_i = L_i$ for $i \in \{1,2,3\}$, and $L'_4 = \{x_1, x_2, \cdots, x_k\}$. If u,v are two non subdivision vertices of $S(G(e;k))$, then clearly $r(u|\Pi') \neq r(v|\Pi')$ since $r(u|\Pi) \neq r(v|\Pi)$. If u,v are two subdivision vertices of $S(G(e;k))$, then u and v are distinguished by L'_1 or L'_2, so $r(u|\Pi') \neq r(v|\Pi')$. This implies that Π' is a resolving partition of $S(G(e;k))$. Thus $pd(S(G(e;k))) \leq pd(G)+1 = 4$. As a consequence, $pd(S(G(e;k))) = pd(G)+1$. □

Lemma 3.5. *Let $G \simeq C(m;r)$, $r \geq 3$ and $m = r+1$. If e is a pendant edge of G, then $pd(S(G(e;k))) = pd(G) - 1$.*

Proof. From [5], we know that $pd(G) = r+1$. Without loss of generality, let $e = v_1w_{12}$. We will show that $pd(S(G(e;k))) = pd(G) - 1 = r$. Since $m \geq 2$, there is a vertex having r leaves, by Corollary 2.2, we have that $pd(S(G(e;k)) \geq r$.

Let $\Pi = \{L_1, L_2, \cdots, L_r\}$ be a partition of $V(S(G(e;k)))$ where $L_1 = \{v_1, v_m, w_{11}, w_{21}, \cdots, w_{m1}, x_1\}$, $L_2 = \{v_2, w_{12}, w_{13}, w_{22}, w_{32}, \cdots, w_{m2}, x_2, x_3, \cdots, x_k\}$, $L_i = \{v_i, w_{1(i+1)}, w_{2i}, \cdots, w_{mi}\}$ for $i \in [3, r-1]$, and $L_r = \{v_r, w_{2r}, w_{3r}, \cdots, w_{mr}\}$. Let u, v be two distinct vertices of $S(G(e;k))$, we will show that $r(u|\Pi') \neq (v|\Pi')$.

Let u and v be two non subdivision vertices. If u, v are backbone vertices, then clearly $r(u|\Pi) \neq r(v|\Pi)$ since $u = v_1$ and $v = v_m$ are distinguished by L_r. Let u and v be two leaves. If $u = w_{tj}$ and $v = w_{sj}$, for some $s, t \in [2, m]$, then u and v are distinguished by L_t. If $u = w_{1j}$, then u and v are distinguished by L_r. Therefore, we have $r(u|\Pi) \neq r(v|\Pi)$. If u is a backbone vertex and v is a leaf, then $r(u|\Pi)$ contains at least two 1s and $r(v|\Pi)$ contains at most one 1. Therefore $r(u|\Pi) \neq r(v|\Pi)$.

Now, let u be a subdivision vertex and v be any other vertex. Since u and v are distinguished by L_r, we have $r(u|\Pi) \neq r(v|\Pi)$. This implies that $pd(S(G(e;k))) = r = pd(G) - 1$. □

Lemma 3.6. *Let $G \simeq C(m;r)$, $r \geq 3$ and $2 \leq m \leq r$. If e is a pendant edge of G, then $pd(S(G(e;k))) = pd(G)$, for $k \geq 1$.*

Proof. By [5], we have that $pd(G) = r$ with $\Pi = \{L_1, L_2, \cdots, L_r\}$ is a resolving partition of G, where $L_i = \begin{cases} \{v_i, w_{1i}, w_{2i}, \cdots, w_{mi}\} & \text{if } 1 \leq i \leq m, \\ \{w_{1i}, w_{2i}, \cdots, w_{mi}\} & \text{if } m+1 \leq i \leq r. \end{cases}$
Without loss of generality, let $e = v_1w_{12}$. Since there is a vertex having r leaves, by Corollary 2.2, $pd(S(G(e;k))) \geq r$.

Let $\Pi' = \{L'_1, L'_2, \cdots, L'_r\}$ be a partition of $V(S(G(e;k)))$ where $L'_i = L_i$, for $i \in [1, r]$ and $i \neq 2$, $L'_2 = L_2 \cup \{x_1, x_2, \cdots, x_k\}$. Let u, v be any two vertices of $S(G(e;k))$ in the same partition class. We will show that $r(u|\Pi) \neq r(v|\Pi)$.

If u, v are two non subdivision vertices, then $r(u|\Pi') \neq r(v|\Pi')$ since $r(u|\Pi) \neq r(v|\Pi)$. If u is a subdivision vertex and v is any other vertex of $S(G(e;k))$, then we have $r(u|\Pi') \neq r(v|\Pi')$, since u and v are distinguished by L'_1 or L'_3. As a consequence, $pd(S(G(e;k))) \leq r$; and so $pd(S(G(e;k))) = pd(G)$. □

Lemma 3.7. *Let $G \simeq C(m;r)$, $r \geq 4$ and $2 \leq m \leq r$. If e is a backbone edge of G, then $pd(S(G(e;k))) = pd(G)$, for $k \geq 1$.*

Proof. From [5], we know that $pd(G) = r$ with $\Pi = \{L_1, L_2, \cdots, L_r\}$ is a resolving partition of G where $L_i = \begin{cases} \{v_i, w_{1i}, w_{2i}, \cdots, w_{mi}\} & \text{if } 1 \leq i \leq m, \\ \{w_{1i}, w_{2i}, \cdots, w_{mi}\} & \text{if } m+1 \leq i \leq r. \end{cases}$

Without loss of generality, let $e = v_1v_2$. Since vertex v_1 have r leaves, by Corollary 2.2, $pd(S(G(e;k))) \geq r$.

Now, let $\Pi' = \{L'_1, L'_2, \cdots, L'_r\}$ be a partition of $S(G(e;k))$. If $k = 1$, then define $L'_i = L_i$ for all $i \neq 3$ and $L'_3 = L_3 \cup \{x_1\}$. If $k = 2$, then define $L'_i = L_i$ for all $i \neq 3, 4$ and $L'_3 = L_3 \cup \{x_1\}$, $L'_4 = L_4 \cup \{x_2\}$. If $k \geq 3$, then $L'_i = L_i$ for all $i \neq 1, 2, 3$ and $L'_1 = L_1 \cup \{x_k\}$, $L'_2 = L_2 \cup \{x_1\}$ and $L'_3 = L_3 \cup \{x_i | 2 \leq i \leq k-1\}$.

For $k = 1, 2, 3$. Let u and v be two non subdivision vertices. Clearly that $r(u|\Pi') \neq r(v|\Pi')$ since $r(u|\Pi) \neq r(v|\Pi)$. Let u be a subdivision vertex and v be any vertex $S(G(e;k))$. Since $r(u|\Pi')$ contains exactly two 1s and $r(v|\Pi')$ contains $r-1$ or at most one 1, we have $r(u|\Pi') \neq r(v|\Pi')$. This implies that Π' is a resolving partition of $S(G(e;k))$.

For $k \geq 4$. Let u, v be two distinct subdivision vertices. Since u, v are distinguished by L'_1 or L'_2, so we obtain $r(u|\Pi') \neq r(v|\Pi')$. Now, let u be a subdivision vertex and v be a non subdivision vertex. If $u \in \{x_1, x_k\}$, then $r(u|\Pi')$ contains exactly two 1s and $r(v|\Pi')$ contains $r-1$ or at most one 1s, so we obtain $r(u|\Pi') \neq r(v|\Pi')$. If $u = x_i$ for some $i \in [2, k-1]$, then u and v are distinguished by L'_r so $r(u|\Pi') \neq r(v|\Pi')$. If u, v are non subdivision vertices, then $r(u|\Pi') \neq r(v|\Pi')$ since $r(u|\Pi) \neq r(v|\Pi)$. Therefore, Π' is a resolving partition of $S(G(e;k))$ and $pd(S(G(e;k))) = pd(G) = r$. □

Theorem 3.8. *Let $G \simeq C(m;r)$, $r \geq 3$, $m \geq 2$ and $e \in E(G)$. Then,*

$$pd(S(G(e;k))) = \begin{cases} pd(G)+1 & \text{if } e \text{ is a backbone edge and} \\ & \quad ((r{=}3,\ m{=}3 \text{ and } k \geq 1) \text{ or } (r{=}3, m{=}2 \text{ and } k{=}4)), \\ pd(G)-1 & \text{if } e \text{ is a pendant edge and } m{=}r{+}1, \\ pd(G) & \text{otherwise.} \end{cases}$$

Proof. We consider the following two cases:
Case 1. e is a backbone edge of G.

For $r = 3$, $m = 2$. By Lemma 3.3, we have $pd(S(G(e;k))) = pd(G) + 1$ for $k = 4$ and $pd(S(G(e;k))) = pd(G)$ for $k \neq 4$. For $r = 3$ and $m = 3$, by Lemma 3.4, $pd(S(G(e;k))) = pd(G) + 1$ for $k \geq 1$. For $r \geq 4$, $m \leq r$, by Lemma 3.7, we have $pd(S(G(e;k))) = pd(G) = r$.

For $r \geq 3$, $m \geq r+1$. By [5], we have $pd(G) = r+1$, with $\Pi = \{L_1, L_2, \cdots, L_r, L_{r+1}\}$ is a resolving partition of G where $L_1 = \{w_{11}, w_{21}, \cdots, w_{m1}, v_1, \cdots, v_{m-1}\}$, $L_{r+1} = \{v_m\}$ and $L_j = \{w_{ij} | 2 \leq i \leq m\}$ for $2 \leq j \leq r$. We will show that $pd(S(G(e;k))) \geq r+1$. Let Π' be a resolving partition of $S(G(e;k))$. For a contradiction, assume that $pd(S(G(e;k))) = r$. Since each $v_i \in [1, m]$ has r leaves belonging to distinct classes of Π' and $m \geq r+1$, then there exist v_a and v_b, $a \neq b$ such that $r(v_a|\Pi') = r(v_b|\Pi')$, a contradiction. Thus, $pd(S(G(e;k))) \geq r+1$.

Now, let $\Pi' = \{L'_1, L'_2, \cdots, L'_{r+1}\}$ be a partition of $V(S(G(e;k)))$ where $L'_i = L_i$ for all $i \in [1, r+1]$ and $i \neq 1$, $L'_1 = L_1 \cup \{x_i | 1 \leq i \leq k\}$. Let u, v be any two distinct vertices in the same class of Π'. If $u \neq v_i$ and $v \neq w_{(i+1)1}$, $i \in [1, m-1]$, then they are distinguished by L'_{r+1}. So, we have $r(u|\Pi') \neq r(v|\Pi')$. If $u = v_i$ and $v = w_{(i+1)1}$, then they are distinguished by L'_2 so we have $r(u|\Pi') \neq r(v|\Pi')$. This implies Π' is a resolving of $S(G(e;k))$. Thus, $pd(S(G(e;k))) = r + 1 = pd(G)$.

Case 2. e is a pendant edge of G.

For $r \geq 3$, $m \leq r$. By Lemma 3.6, we have $pd(S(G(e;k))) = pd(G) = r$. For $r \geq 3$, $m = r + 1$. By Lemma 3.5, we have $pd(S(G(e;k))) = r = pd(G) - 1$.

For $r \geq 3$ and $m \geq r + 2$. By [5], we have $pd(G) = r + 1$ with $\Pi = \{L_1, L_2, \cdots, L_r, L_{r+1}\}$ is a resolving partition of G, where $L_1 = \{v_1, v_2, \cdots, v_{m-1}, w_{11}, w_{21}, \cdots, w_{r1}\}$, $L_i = \{w_{1i}, w_{2i}, \cdots, w_{mi}\}$ for $2 \leq i \leq r$, and $L_{r+1} = \{v_m\}$.

We will show that $pd(S(G(e;k)) \geq pd(G) = r + 1$. Without loss of generality, let $e = v_1 w_{12}$. For a contradiction, assume that $pd(S(G(e;k))) = r$. Let Π' be a resolving partition of $S(G(e;k))$. Since each $v_i \in [2, m]$ has r leaves belonging to distinct classes of Π' and $m \geq r + 2$, there are v_a and v_b with $a \neq b \in [2, m]$, so we have $r(v_a|\Pi') = r(v_b|\Pi')$. Thus, $pd(S(G(e;k))) \geq r + 1$.

Now, let $\Pi' = \{L'_1, L'_2, \cdots, L'_r, L_{r+1}\}$ be a partition of $S(G(e;k))$ where $L'_i = L_i$, for $i \neq 2$ and $L'_2 = L_2 \cup \{x_1, x_2, \cdots, x_k\}$. Let u and v be two distinct vertices of $S(G(e;k))$. If $u \neq v_1$ and $v \neq w_{(i+1)1}$, $i \in [1, m-1]$, then they are distinguished by L'_{r+1}. So, we have $r(u|\Pi') \neq r(v|\Pi')$. If $u = v_1$ and $v = w_{(i+1)1}$, $i \in [1, m-1]$, then they are distinguished by L'_2. This implies that $pd(S(G(e;k))) \leq r + 1$. Therefore, $pd(S(G(e;k))) = pd(G) = r + 1$. □

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