The Resolving Graph of Amalgamation of Cycles

H. Iswadi[1,2], E.T. Baskoro[1], A.N.M. Salman[1], R. Simanjuntak[1]

[1]Combinatorial Mathematics Research Division
Faculty of Mathematics and Natural Science, Institut Teknologi Bandung
Jalan Ganesha 10 Bandung 40132, Indonesia.
[2]Department of MIPA, Gedung TG lantai 6, Universitas Surabaya,
Jalan Raya Kalirungkut Surabaya 60292, Indonesia.
hazrul_iswadi@ubaya.ac.id, {ebaskoro, msalman, rino}@math.itb.ac.id

Abstract

For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a connected graph G, the representation of v with respect to W is the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ where $d(x, y)$ represents the distance between the vertices x and y. The set W is called a resolving set for G if every vertex of G has a distinct representation. A resolving set containing a minimum number of vertices is called a basis for G. The dimension of G, denoted by $\dim(G)$, is the number of vertices in a basis of G. A resolving set W of G is connected if the subgraph $\langle W \rangle$ induced by W is a nontrivial connected subgraph of G. The connected resolving number is the minimum cardinality of a connected resolving set in a graph G, denoted by $\mathrm{cr}(G)$. A *cr-set* of G is a connected resolving set with cardinality $\mathrm{cr}(G)$. A connected graph H is a *resolving graph* if there is a graph G with a cr-*set* W such that $\langle W \rangle = H$. Let $\{G_i\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{oi} called a terminal. The amalgamation $\mathrm{Amal}\{G_i, v_{oi}\}$ is formed by taking of all the G_i's and identifying their terminals. In this paper, we determine the connected resolving number and characterize the resolving graphs of amalgamation of cycles.

1 Introduction

In this paper we consider finite, simple, and connected graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a further reference please see Chartrand and Lesniak [3].

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. For an ordered set $W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G)$, we refer to the ordered k-tuple $r(v|W) = (d(v,w_1), d(v,w_2), \cdots, d(v,w_k))$ as the (*metric*) *representation of* $v \in V(G)$ *with respect to* W. The set W is called a *resolving set* for G if $r(u|W) = r(v|W)$ implies $u = v$ for all $u, v \in G$. A resolving set with minimum cardinality is called a *minimum resolving set* or a *basis*. The *metric dimension* of a graph G, $\dim(G)$, is the number of vertices in a basis for G. To determine whether W is a resolving set for G, we only need to investigate the representations of the vertices in $V(G)\backslash W$, since the representation of each $w_i \in W$ has '0' in the ith-ordinate; and so it is always unique.

The first papers discussing the notion of a (minimum) resolving set were written by Slater in [19] and [20]. Slater introduced the concept of a resolving set for a connected graph G under the term *location set*. He called the cardinality of a minimum resolving set by the *location number* of G. Independently, Harary and Melter [8] introduced the same concept, but used the term *metric dimension* instead.

In general, finding a resolving set for arbitrary graph is a difficult problem. In [7], it is proved that the problem of computing the metric dimension for general graphs is NP-complete. Thus, researchers in this area often studied the metric dimension for particular classes of graphs or characterized graphs having certain metric dimension. Some results on the joint graph and cartesian product graph have been obtained by Caceres et al. [1], Khuller et al. [13], and Chartrand [4]. Iswadi et.al obtained some results on the corona product of graphs [10, 11]. Suhadi et al. obtained some results on the decomposition product of graphs [23]. Iswadi et al. determined the metric dimension of antipodal and pendant free graph [12]. Suhadi et al. found some results on the metric dimension of some type of regular graphs [21, 22]. And, Chartrand et al. [4] has characterized all graphs having metric dimensions $1, n-1$, and $n-2$. They also determined the metric dimensions of some well known families of graphs such as paths, cycles, complete graphs, and trees.

The study of finding resolving sets of graphs can also be conducted by considering particular restrictions for the resolving sets. One of the restrictions is connectivity; in [16] Saenpholphat and Zhang introduced the concept of connected resolvability. They defined the following terms. A resolving set W of G is *connected* if the subgraph $\langle W \rangle$ induced by W is a nontrivial connected subgraph of G. *The connected resolving number* $\mathrm{cr}(G)$ is the minimum cardinality of a connected resolving set in G. A *cr-set* of G is a connected resolving set with cardinality $\mathrm{cr}(G)$. Since every connected resolving set is a resolving set, then $\dim(G) \leq \mathrm{cr}(G)$ for any connected graph. A connected graph H is a *resolving graph* if there is a graph G with a cr-*set* W such that $\langle W \rangle = H$. Additionally, they observed the following.

Observation A [16] *Let G be a graph and $U \subseteq V(G)$. If U contains a resolving set of G as its subset, then U is also a resolving set of G.*

Observation B [16] *Let G be a connected graph. Then $dim(G) = cr(G)$ if and only if G contains a connected basis.*

Further properties of connected resolving sets in a graph and its relation with the basis of the graph can be found in [5, 15, 17] and [18].

The following identification graph $G = G[G_1, G_2, v_1, v_2]$ is defined in [14].

Definition C *Let G_1 and G_2 be the non trivial connected graphs where $v_1 \in G_1$ and $v_2 \in G_2$. An identification graph $G = G[G_1, G_2, v_1, v_2]$ is obtained from G_1 and G_2 by identifying v_1 and v_2 such that $v_1 = v_2$ in G.*

Poisson et.al. [14] determined the lower and upper bound of metric dimension of $G[G_1, G_2, v_1, v_2]$ in terms of $dim(G_1)$ and $dim(G_2)$ as stated in the following theorems.

Theorem D *Let G_1 and G_2 be the non trivial connected graphs with $v_1 \in G_1$ and $v_2 \in G_2$ and let $G = G[G_1, G_2, v_1, v_2]$. Then*

$$dim(G) \geq dim(G_1) + dim(G_2) - 2.$$

For the upper bound, we define an equivalence class and binary function first. For a set W of vertices of G, define a relation on $V(G)$ with respect to W by uRv if there exists $a \in \mathbb{Z}$ such that $r(v|W) = r(u|W) + (a, a, \cdots, a)$. It is easy to check that R is an equivalence relation on $V(G)$. Let $[u]_W$ denote the equivalence class of u with respect to W. Then

$$v \in [v]_W \text{ if and only if } r(v|W) \ = r(u|W) + (a, a, \cdots, a)$$

for some $a \in \mathbb{Z}$. For a non trivial connected graph G, define a binary function $f_G : V(G) \to \mathbb{Z}$ with

$$f_G(v) = \begin{cases} dim(G), & \text{if } v \text{ is not a basis vertex of } G; \\ dim(G) - 1, & \text{otherwise.} \end{cases}$$

Theorem E *Let G_1 and G_2 be the non trivial connected graphs with $v_1 \in G_1$ and $v_2 \in G_2$ and let $G = G[G_1, G_2, v_1, v_2]$. Suppose that G_1 contains a resolving set W_1 such that $[v_1]_{W_1} = \{v_1\}$. Then*

$$\begin{aligned} dim(G) &\leq |W_1| + f_{G_2}(v_2) \\ &= \begin{cases} |W_1| + dim(G_2), & \textit{if } v_2 \textit{ is not a basis vertex of } G_2; \\ |W_1| + dim(G_2) - 1, & \textit{otherwise.} \end{cases} \end{aligned}$$

In particular, if W_1 is a basis for G_1, then

$$dim(G) \leq \begin{cases} dim(G_1) + dim(G_2), & \textit{if } v_2 \textit{ is not a basis vertex of } G_2; \\ dim(G_1) + dim(G_2) - 1, & \textit{otherwise.} \end{cases}$$

In this paper, we determine the metric dimension, the connected resolving number, and characterize the resolving graphs of amalgamation of cycles.

2 Amalgamation of Cycles

The following definition of amalgamation of graphs is taken from [2].

Definition F *Let $\{G_i\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{oi} called a terminal. The amalgamation $Amal\{G_i, v_{oi}\}$ is formed by taking of all the G_i's and identifying their terminals.*

Definition F is a generalization of Definition C. If the collection of graphs in Definition F only consist two graphs then we get Definition F is exactly Definition C.

We could consider amalgamation of cycles; that is Amal$\{G_i, v_{oi}\}$ where $G_i = C_n$ for all i. In this particular amalgamation, the choice of vertex v_{oi} is irrelevant. So, for simplification, we can denote this amalgamation by $(C_n)_t$, where t denotes the number of cycle C_n. For $t = 1$, the graph $(C_n)_1$ is the cycle C_n. For $n = 3$, the graph $(C_3)_t$ is called the friendship graph or the Dutch t-windmill [6].

In this paper, we consider a generalization of $(C_n)_t$ where the cycles under consideration may be of different lenghts. We denote this amalgamation by Amal$\{C_{n_i}\}$, $1 \leq i \leq t$, $t \geq 2$. We call every C_{n_i} (including the terminal) in Amal$\{C_{n_i}\}$ as a *leaf* and a path P_{n_i-1} obtained from C_{n_i} by deleting the terminal as *a nonterminal path.*

Throughout this paper, we will follow the following notations and labels for cycles, nonterminal paths, and vertices in Amal$\{C_{n_i}\}$. For odd n_i, $n_i = 2k_i + 1$, $k_i \geq 1$ and for the terminal vertex x, we label all vertices in each leaf C_{n_i} such that

$$C_{n_i} = xv_1^i v_2^i \cdots v_{k_i}^i w_{k_i}^i w_{k_i-1}^i \cdots w_1^i x,$$

this will give the nonterminal path

$$P_{n_i-1} = v_1^i v_2^i \cdots v_{k_i}^i w_{k_i}^i w_{k_i-1}^i \cdots w_1^i.$$

For even n_i, $n_i = 2k_i + 2$, $k_i \geq 1$, and for the terminal vertex x, we define the labels of all vertices in each leaf C_{n_i} as follow

$$C_{n_i} = xv_1^i v_2^i \cdots v_{k_i}^i u^i w_{k_i}^i w_{k_i-1}^i \cdots w_1^i x,$$

which leads to the following labeling of the nonterminal path

$$P_{n_i-1} = v_1^i v_2^i \cdots v_{k_i}^i u^i w_{k_i}^i w_{k_i-1}^i \cdots w_1^i.$$

Iswadi et. al [10] characterized the resolving set and determined the metric dimension of amalgamation of cycles Amal$\{C_{n_i}\}$ as stated in the following lemma and theorem.

Lemma G *Let S be a resolving set of Amal$\{C_{n_i}\}$. Then $|P_{n_i-1} \cap S| \geq 1$, for each i.*

Theorem H *If Amal$\{C_{n_i}\}$ is an amalgamation of t cycles that consists of t_1 number of odd cycles and t_2 number of even cycles, then*

$$dim(Amal\{C_{n_i}\}) = \begin{cases} t_1, & t_2 = 0; \\ t_1 + 2t_2 - 1, & otherwise. \end{cases}$$

One of the natural questions we could pose after proving Theorem H is: Are there any bases other than the bases we constructed in the proof of Theorem H? Let the number of different basis of G be denoted by $\sharp G$. Iswadi et. al [9] also determined the number of different basis of Amal$\{C_{n_i}\}$ as stated in the following theorem.

Theorem I *If Amal$\{C_{n_i}\}$ is an amalgamation of t cycles that consists of t_1 number of odd cycles and t_2 number of even cycles, then*

$$\sharp Amal\{C_{n_i}\} = \begin{cases} 2^{t-1}\left(\sum_{i=1}^{t}(n_i-1)-2\right), & t_2 = 0; \\ 2^{t_1}\left(n_{t_1+1}-1\right)\prod_{j=t_1+2}^{t}\left(C(n_j-1,2)-2C(k_j,2)\right), & otherwise. \end{cases}$$

where $C(b,a)$ is the total number of combinations of b objects taken a.

3 Resolving graph of amalgamation of cycles

By considering all bases of amalgamation of cycles identified in [9], we will show that Amal$\{C_{n_i}\}$ contains no connected basis.

Theorem 1. *Amal$\{C_{n_i}\}$ has no connected basis.*

Proof. Let x be a terminal vertex of Amal$\{C_{n_i}\}$. By direct inspection to all of bases B of Amal$\{C_{n_i}\}$, we have $x \notin B$. It is easy to show that x is a cut-vertex of Amal$\{C_{n_i}\}$. Since $x \notin B$, where B is a basis of Amal$\{C_{n_i}\}$, x is a cut-vertex of Amal$\{C_{n_i}\}$, and every nonterminal path P_{n_i-1} must contain at least one vertex of every resolving set of Amal$\{C_{n_i}\}$ then the subgraph $\langle B\rangle$ must be disconnected. □

Next, we will determine the connected resolving number and the resolving graph of an amalgamation of cycles. Since every basis B of Amal$\{C_{n_i}\}$ is unconnected then we must choose a resolving set other than a basis of Amal$\{C_{n_i}\}$ to form a connected resolving graph. From the proof of Theorem 1, x must be contained in any connected resolving set. Hence, the connected resolving set of Amal$\{C_{n_i}\}$ is

$$W = S \cup \{x\},$$

where $S \subseteq \bigcup_{i=1}^{t} P_{n_i-1}$.

Lemma 1. *Let S be a connected resolving set and P_{n_i-1} be a nonterminal path of Amal$\{C_{n_i}\}$ with $n_i \geq 4$. If $|P_{n_i-1} \cap S|$ = 1 for some i then $P_{n_i-1} \cap S$ = $\{v_1^i\}$ or $\{w_1^i\}$.*

Proof. Let x be a terminal vertex of Amal$\{C_{n_i}\}$. Let $u \in P_{n_i-1} \cap S$. Suppose that $u \neq v_1^i$ and w_1^i. Since $n_i \geq 4$ then u will be one of v_a^i with $2 \leq a \leq k_i$, or w_a^i with $2 \leq a \leq k_i$, or u^i. For each case, u is not adjacent to x, which is a contradiction with S being a connected resolving set of Amal$\{C_{n_i}\}$. □

Lemma 2. *Let S be a connected resolving set of Amal$\{C_{n_i}\}$. Let P_{n_i-1} and P_{n_j-1} be a pair of nonterminal paths of Amal$\{C_{n_i}\}$ with n_i, $n_j \geq 4$. Then $|(P_{n_i-1} \cup P_{n_j-1}) \cap S| \geq 3$.*

Proof. Let x be a terminal vertex of Amal$\{C_{n_i}\}$. Since S is a resolving set of Amal$\{C_{n_i}\}$, by using Lemma H, we have $|(P_{n_i-1} \cup P_{n_j-1}) \cap S| \geq 2$. Suppose that $|(P_{n_i-1} \cup P_{n_j-1}) \cap S| = 2$. Let $u \in P_{n_i-1} \cap S$ and $v \in P_{n_j-1} \cap S$. By using Lemma 1, $u = v_1^i$ or w_1^i and $v = v_1^j$ or w_1^j. Without loss of generality, let $u = v_1^i$ and $v = v_1^j$. Since n_i, $n_j \geq 4$, then w_1^i and w_1^j have the same distance to every $z \in S$, $d(w_1^i, z) = d(w_1^j, z)$, a contradiction with S being a connected resolving set in Amal$\{C_{n_i}\}$. Therefore, $|(P_{n_i-1} \cup P_{n_j-1}) \cap S|$ ≥ 3. □

Theorem 2. *If Amal$\{C_{n_i}\}$ is the amalgamation of r cycles that consists of r_1 number of cycles C_3 and r_2 number of cycles C_{n_i} with $n_i \geq 4$, then*

$$cr(Amal\{C_{n_i}\}) = \begin{cases} r_1 + 1, & r_2 = 0; \\ r_1 + 2r_2, & \textit{otherwise.} \end{cases}$$

Proof. Let B be a connected resolving set with minimum cardinality and x a terminal vertex of Amal$\{C_{n_i}\}$. Let $S \subseteq \bigcup_{i=1}^{r} P_{n_i-1}$ be a set of Amal$\{C_{n_i}\}$ and $W = S \cup \{x\}$. We label all the leaves C_{n_i} of Amal$\{C_{n_i}\}$ in such a way that C_{n_i}'s with $1 \leq i \leq r_1$, are C_3 and C_{n_j}'s, with $r_1 + 1 \leq j \leq r_1 + r_2 = r$, are cycles with length more than three.

Case 1. For $r_2 = 0$. Since W is a resolving set of Amal$\{C_{n_i}\}$ then, by using Lemma H, we have $|\bigcup_{i=1}^{r}(P_{n_i-1} \cap W)| = |P_{n_1-1} \cap W| + \cdots + |P_{n_2-1} \cap W| \geq r_1$. Since $x \in W$ and $x \notin \bigcup_{i=1}^{r} P_{n_i-1}$ then $|W| \geq r_1 + 1$. Therefore, $|B| \geq r_1 + 1$. Choose a set $W = \bigcup_{i=1}^{r_1} S_i \cup \{x\}$, where $S_i = \{w_1^i\}$ with $1 \leq i \leq r_1$. The representations of other vertices in Amal$\{C_{n_i}\}$ by W are

$$r(w_1^i|W) = (2, 2, \underbrace{1}_{coord.\ of\ S_i}, 2, 2, 1) \text{ with } 1 \leq i \leq r_1 = r.$$

Therefore, all vertices of Amal$\{C_{n_i}\}$ have distinct representations by W. Hence, W is a connected resolving set of Amal$\{C_{n_i}\}$. Since B is a minimum connected resolving set of Amal$\{C_{n_i}\}$ then $|B| \leq r_1 + 1$. So, we have $|B| = r_1 + 1$.

Case 2. For $r_2 \geq 1$. By using Lemmas H and 2, we have $|\bigcup_{i=1}^{r} P_{n_i-1} \cap W| = |P_{n_1-1} \cap W| + \cdots + |P_{n_2-1} \cap W| \geq r_1 + 2r_2 - 1$. By a similar argument as in Case 1, since $x \in W$ and $x \notin \bigcup_{i=1}^{r} P_{n_i-1}$ then $W \geq r_1 + 2r_2$. Therefore, we have $|B| \geq r_1 + 2r_2$. Next, we will show that $|B| \leq r_1 + 2r_2$. Choose a set $W = \bigcup_{i=1}^{r} S_i \cup \{x\}$ with

$$\begin{aligned} S_i &= \{w_1^i\}, \text{ with } 1 \leq i \leq r_1, \\ S_{r_1+1} &= \{w_1^{r_1+1}\}, \\ S_j &= \{v_1^j, w_1^j\}, \text{ with } r_1 + 2 \leq j \leq r. \end{aligned}$$

The representations of the other vertices of Amal$\{C_{n_i}\}$ by W are as follow.

For $1 \leq i \leq r_1$,

$$r(v_1^i|W) = (2, \cdots, 2, \underbrace{1}_{coord.\ of\ S_i}, 2, \cdots, 2, 1) \text{ with } 1 \leq i \leq r_1 + 1.$$

For $j = r_1 + 1$, $n_j = 2k_j + 1$ with $k_j \geq 2$,

$$\begin{aligned} r(v_{k_j}^{r_1+1}|W) &= (k_j + 1, \cdots, k_j + 1, \underbrace{k_j}_{coord.\ of\ S_{r_1+1}}, k_j + 1, \cdots, k_j + 1, k_j), \\ r(v_l^{r_1+1}|W) &= (l + 1, \cdots, l + 1, \underbrace{l + 1}_{coord.\ of\ S_{r_1+1}}, l + 1, \cdots, l + 1, l) \\ &\quad \text{with } 1 \leq l \leq k_j - 1, \text{ and} \\ r(w_l^{r_1+1}|W) &= (l + 1, \cdots, l + 1, \underbrace{l - 1}_{coord.\ of\ S_{r_1+1}}, l + 1, \cdots, l + 1, l) \\ &\quad \text{with } 2 \leq l \leq k_j. \end{aligned}$$

For $j = r_1 + 1$, $n_j = 2k_j + 2$ with $k_j \geq 2$,

$$
\begin{aligned}
r(u^{r_1+1}|W) &= (k_j+2, \cdots, k_j+2, \underbrace{k_j}_{coord.\ of\ S_{r_1+1}}, k_j+2, \cdots, k_j+2, k_j+1), \\
r(v_l^{r_1+1}|W) &= (l+1, \cdots, l+1, \underbrace{l+1}_{coord.\ of\ S_{r_1+1}}, l+1, \cdots, l+1, l) \\
&\quad \text{with } 1 \leq l \leq k_j, \text{ and} \\
r(w_l^{r_1+1}|W) &= (l+1, \cdots, l+1, \underbrace{l-1}_{coord.\ of\ S_{r_1+1}}, l+1, \cdots, l+1, l) \\
&\quad \text{with } 2 \leq l \leq k_j.
\end{aligned}
$$

For $r_1 + 2 \leq j \leq r$, $n_j = 2k_j + 1$ with $k_j \geq 2$,

$$
\begin{aligned}
r(v_{k_j}^j|W) &= (k_j+1, \cdots, k_j+1, \underbrace{k_j-1, k_j}_{coord.\ of\ S_j}, k_j+1, \cdots, k_j+1, k_j), \\
r(w_{k_j}^j|W) &= (k_j+1, \cdots, k_j+1, \underbrace{k_j, k_j-1}_{coord.\ of\ S_j}, k_j+1, \cdots, k_j+1, k_j), \\
r(v_l^j|W) &= (l+1, \cdots, l+1, \underbrace{l-1, l+1}_{coord.\ by\ S_j}, l+1, \cdots, l+1, l) \\
&\quad \text{with } 1 \leq l \leq k_j - 1, \text{ and} \\
r(w_l^j|W) &= (l+1, \cdots, l+1, \underbrace{l+1, l-1}_{coord.\ of\ S_j}, l+1, \cdots, l+1, l) \\
&\quad \text{with } 1 \leq l \leq k_j - 1.
\end{aligned}
$$

For $r_1 + 2 \leq j \leq r$, $n_j = 2k_j + 2$ with $k_j \geq 1$,

$$
\begin{aligned}
r(u^j|W) &= (k_j+2, \cdots, k_j+2, \underbrace{k_j, k_j}_{coord.\ of\ S_j}, k_j+2, \cdots, k_j+2, k_j+1), \\
r(v_l^j|W) &= (l+1, \cdots, l+1, \underbrace{l-1, l+1}_{coord.\ of\ S_j}, l+1, \cdots, l+1, l) \\
&\quad \text{with } 1 \leq l \leq k_j, \text{ and} \\
r(w_l^j|W) &= (l+1, \cdots, l+1, \underbrace{l+1, l-1}_{coord.\ of\ S_j}, l+1, \cdots, l+1, l) \\
&\quad \text{with } 1 \leq l \leq k_j.
\end{aligned}
$$

Thus, all vertices of Amal$\{C_{n_i}\}$ have distinct representations by W. Therefore, W is a connected resolving set of Amal$\{C_{n_i}\}$ and so, $|B| \leq r_1 + 2r_2$, which complete the proof. $\square$

A graph G' is called *a subdivision* of a graph G if one or more vertices of degree 2 are inserted into one or more edges of G. In the following theorem, we state all possible resolving graphs of Amal$\{C_{n_i}\}$.

Theorem 3. *The resolving graph H of Amal$\{C_{n_i}\}$ is either a path, a star, or a subdivision of a star.*

Proof. Let x be the terminal vertex of Amal$\{C_{n_i}\}$. Assume Amal$\{C_{n_i}\}$ consists of r cycles with r_1 number of cycles C_3 and r_2 number of cycles C_{n_i} with $n_i \geq 4$. Let W be a connected resolving set with minimum cardinality of Amal$\{C_{n_i}\}$. Let P_{n_i-1} and P_{n_j-1} be the two nonterminal paths of Amal$\{C_{n_i}\}$.

Case 1. For $r = 2$, there are three subcases; when $r_1 = 2, r_2 = 0$ or $r_1 = 1, r_2 = 1$ or $r_1 = 0, r_2 = 2$. By using Lemma H, Lemma 1, and Theorem 2, for all of these subcases, $W = \{x, a, b\}$, where $a = v_1^1$ or w_1^1 and $b = v_1^2$ or w_1^2. The subgraph $\langle W \rangle$ is a path P_3 which contains 3 vertices.

Case 2. For $r \geq 3$, there are two subcases.
Claim: If $n_i > 4$ and $|P_{n_i-1} \cap W| = 2$ then $P_{n_i-1} \cap W = \{v_1^i, w_1^i\}$. By using Theorem 2, there is one leaf C_{n_j} such that $n_j \geq 4$ and $|P_{n_j-1} \cap W| = 1$. By using symmetry property, assume that $P_{n_i-1} \cap W = \{v_1^i, v_2^i\}$ and $P_{n_j-1} \cap W = \{v_1^j\}$. Hence, $d(w_1^i, z) = d(w_1^j, z)$ for all $z \in W$, a contradiction with W being a resolving set. *Subcase 2.1,* there is no leaf C_{n_i} with $n_i = 4$. By using Lemma H, Theorem 2, and the previous claim, $W = \{x, a_1, \cdots, a_{t-1}\}$ where $d(a_i, x) = 1$ with $1 \leq i \leq r - 1$. Hence, the subgraph $\langle W \rangle$ is a star S_{r-1} which contains r vertices. *Subcase 2.2,* there are some leaves C_{n_i} with $n_i = 4$. If $r_2 = 1$, by using similar reasons with Subcase 2.1, then we also have the subgraph $\langle W \rangle$ as a star S_{r-1} which contains r vertices. If $r_2 \geq 2$ and $W = \{x, a_1, \cdots, a_{r-1}\}$ is a connected resolving set with minimum cardinality of Amal$\{C_{n_i}\}$ then $d(a_i, x) = 1$ for all $a_i \in W$ or there are two vertices $a, b \subseteq W$ such that $d(a, x) = 1$ and $d(b, x) = 2$. The previous gives the subgraph $\langle W \rangle$ as a star S_{r-1} which contain r vertices and the last gives the subgraph $\langle W \rangle$ as a subdivision of a star S_m for some m. $\square$

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References

[1] J. Caceres, C. Hernando, M. Mora, M.L. Puertas, I.M. Pelayo, and C. Seara, On the metric dimension of some families of graphs, *Electronic Notes in Discrete Math.*, **22** (2005), 129 – 133.

[2] K. Carlson, Generalized books and C_m-snakes are prime graphs, *Ars Combin.*, **80** (2006), 215-221.

[3] G. Chartrand, and L. Lesniak, *Graphs and Digraphs*, 3rd ed., Chapman and Hall/CRC, 2000.

[4] G. Chartrand, L. Eroh, M.A. Johnson, and O.R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, **105** (2000), 99 - 113.

[5] G. Chartrand and P. Zhang, The theory and appllications of resolvability in graphs: a survey, *Congr. Numer.*, **160** (2003), 47-68.

[6] J.A. Gallian, A dynamic survey of graph labeling, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics* **16** (2009), $\sharp DS6$.

[7] M.R. Garey and D.S. Johnson, *Computers and Intractability: a guide to the theory of NP-completeness*, W.H. Freeman, California, 1979.

[8] F. Harary and R.A. Melter, On the metric dimension of a graph, *Ars. Combin.*, **2** (1976), 191-195.

[9] H. Iswadi, E.T. Baskoro, A.N.M. Salman, and R. Simanjuntak, The metric dimension of amalgamation of cycles, *Far East Journal of Mathematical Science* **41:1** (2010), 19-31.

[10] H. Iswadi, E.T. Baskoro, R. Simanjuntak, and A.N.M. Salman, The metric dimensions of graphs with pendant edges, *J. Combin. Math. Combin. Comput.* **65** (2008), 139-145.

[11] H. Iswadi, H. Assiyatun, E.T Baskoro, and R. Simanjuntak, On the metric dimension of corona product of graphs, preprint.

[12] H. Iswadi, H. Assiyatun, E.T Baskoro, and R. Simanjuntak, Metric dimension of antipodal and pendant-free block-cactus graphs, preprint.

[13] S. Khuller, B. Raghavachari, and A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.*, **70** (1996), 217-229.

[14] C. Poisson and P. Zhang, The metric dimension of unicyclic graphs, *J. Combin. Math. Combin. Comput.*, **40** (2002), 17-32.

[15] V. Saenpholphat and P. Zhang, Some results on connected resolvability in graphs, *Congr. Numer.*, **158** (2002), 5-19.

[16] V. Saenpholphat and P. Zhang, Connected resolvability of graphs, *Czechoslovak Math. J.*, **53:128** (2003), 827-840.

[17] V. Saenpholphat and P. Zhang, Connected resolving sets in graphs, *Ars. Combin.*, **68** (2003), 3-16.

[18] V. Saenpholphat and P. Zhang, Conditional resolvability: a survey, *Internat. J. Math. Math. Sci.*, **38** (2004), 1997-2017.

[19] P.J. Slater, Leaves of trees, *Congr. Numer.*, **14** (1975), 549-559.

[20] P.J. Slater, Dominating and reference sets in graphs, *J. Math. Phys. Sci.*, **22** (1988), 445-455.

[21] S.W. Saputro, E.T. Baskoro, A.N.M. Salman, D. Suprijanto, and M. Bača, The metric dimension of (μ;σ)-regular graph, preprint.

[22] S.W. Saputro, E.T. Baskoro, A.N.M. Salman, D. Suprijanto, and M. Bača,The metric dimension of regular bipartite graphs, *Discrete Mathematics and Theoretical Computer Science*, submitted.

[23] S.W. Saputro, H. Assiyatun, R. Simanjuntak, S. Uttunggadewa, E.T. Baskoro, A.N.M. Salman, and M. Bača, The metric dimension of the composition product of graphs, preprint.