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The metric dimension of the lexicographic product of graphs

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ABSTRACT

A set of vertices W *resolves* a graph G if every vertex is uniquely determined by its coordinate of distances to the vertices in W. The minimum cardinality of a resolving set of G is called the *metric dimension* of G. In this paper, we consider a graph which is obtained by the lexicographic product between two graphs. The *lexicographic product* of graphs G and H, which is denoted by $G \circ H$, is the graph with vertex set $V(G) \times V(H) = \{(a, v) \mid a \in V(G), v \in V(H)\}$, where (a, v) is adjacent to (b, w) whenever $ab \in E(G)$, or $a = b$ and $vw \in E(H)$. We give the general bounds of the metric dimension of a lexicographic product of any connected graph G and an arbitrary graph H. We also show that the bounds are sharp.

1. Introduction

Throughout this paper, all graphs G are finite and simple. We denote by V the vertex set of G and by E the edge set of G. The distance between two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is the length of a shortest $u - v$ path in G. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$. For $v \in V(G)$, a *representation* of v with respect to W is defined as the k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is called a *resolving set* of G if every two distinct vertices $x, y \in V(G)$ satisfy $r(x|W) \neq r(y|W)$. A *basis* of G is a resolving set of G with the minimum cardinality, and the *metric dimension* of G refers to its cardinality and is denoted by $\beta(G)$.

The metric dimension problems were first studied by Harary and Melter [6], and independently by Slater [18,19]. Khuller et al. [11] studied the metric dimension motivated by the robot navigation in a graph space. A resolving set for a graph corresponds to the presence of distinctively labeled "landmark" nodes in the graph. It is assumed that a robot can detect the distance to each node of the landmarks, and hence uniquely determine its location in the graph.

Garey and Johnson [5], and also Khuller et al. [11], showed that determining the metric dimension of an arbitrary graph is an NP-complete problem. However, Chartrand et al. [3] have obtained some results as follows.

Theorem 1 (*[3]*). *Let G be a connected graph of order $n \geq 2$. Then*

1. $\beta(G) = 1$ *if and only if* $G = P_n$.
2. $\beta(G) = n - 1$ *if and only if* $G = K_n$.

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3. *For* $n \geq 3$, $\beta(C_n) = 2$.
4. $\beta(G) = n - 2$ *if and only if* G *is either* $K_{r,s}$ *for* $r, s \geq 1$, *or* $K_r + \overline{K_s}$ *for* $r \geq 1, s \geq 2$, *or* $K_r + (K_1 \cup K_s)$ *for* $r, s \geq 1$.

Many researchers have also considered this problem for certain particular classes of graphs, such as trees [3,6,11], fans [2], wheels [1,2,17], complete n-partite graphs [3,16], unicyclic graphs [14], grids [13], honeycomb networks [12], circulant networks [15], Cayley graphs [4], graphs with pendants [9], amalgamation of cycles [10], and Jahangir graphs [20].

There are also some results of the metric dimension problem for graphs resulting from operations on graphs. We recall that the *joint graph* of G and H, which is denoted by $G+H$, is a graph with $V(G+H) = V(G) \cup V(H)$ with $V(G) \cap V(H) = \emptyset$ and $E(G+H) = E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. Some results on certain joint product graphs have been proved in [1,2,17].

Caceres et al. [2], Khuller et al. [11], and Melter et al. [13] have determined the metric dimension of graphs which are obtained by the Cartesian product of two or more graphs. Some graphs which are constructed by the corona product of two graphs have been studied in [9,8,21]. In this paper, we study the metric dimension of the *lexicographic product* of connected graph G and an arbitrary graph H. We give general bounds of the metric dimension and also show that the bounds are sharp.

2. The main results

The *lexicographic product* of graphs G and H, which is denoted by $G \circ H$ [7], is the graph with vertex set $V(G) \times V(H) = \{(a, v) | a \in V(G), v \in V(H)\}$, where (a, v) is adjacent to (b, w) whenever $ab \in E(G)$, or $a = b$ and $vw \in E(H)$. For any vertex $a \in V(G)$ and $b \in V(H)$, we define the vertex set $H(a) = \{(a, v) | v \in V(H)\}$ and $G(b) = \{(v, b) | v \in V(G)\}$.

Let G be a connected graph with $|V(G)| \geq 2$ and H be an arbitrary graph containing k components $H_1, H_2, \ldots, H_k$ and $|V(H)| \geq 2$. For $a \in V(G)$ and $1 \leq i \leq k$, we define the vertex set $H_i(a) = \{(a, v) \mid v \in V(H_i)\}$. We obtain the following propositions.

Proposition 1. *Let* a *and* b *be two distinct vertices in* G. *Every two different vertices* $x, y \in H(a)$ *satisfy* $d(x, z) = d(y, z)$ *whenever* $z \in H(b)$.

Proof. Let $V(H) = \{h_1, h_2, \ldots, h_{|V(H)|}\}$. Let $x = (a, h_p)$, $y = (a, h_q)$, and $z = (b, h_r)$ where $p, q, r \in \{1, 2, \ldots, |V(H)|\}$ and $p \neq q$. Note that, by the definition of $G \circ H$, every vertex of $H(u)$ is adjacent to every vertex of $H(v)$ for $uv \in E(G)$. Now, for $a \in V(G)$, let u_a be a projection of all vertices of $H(a)$. Let Q be a graph where $V(Q) = \{u_a \mid a \in V(G)\}$ and $u_a u_b \in E(Q)$ whenever $ab \in E(G)$. So, the distance between x and z, $d(x, z)$, in $G \circ H$ is equal to the distance between u_a and u_b, $d(u_a, u_b)$, in Q. Since a vertex y is also projected to u_a, we obtain that $d(y, z) = d(u_a, u_b) = d(x, z)$. □

Proposition 2. *For* $a \in V(G)$ *and* $i, j \in \{1, 2, \ldots, k\}$ *with* $i \neq j$, *every two different vertices* $x, y \in H_j(a)$ *satisfy* $d(x, z) = d(y, z)$ *whenever* $z \in H_i(a)$.

Proof. Let $b \in V(G)$ and $ab \in E(G)$. Since all vertices of $H(a)$ are adjacent to all vertices of $H(b)$, for $w \in H(b)$, we obtain that $d(x, z) = d(x, w) + d(w, z) = 2 = d(y, w) + d(w, z) = d(y, z)$. □

By considering Propositions 1 and 2, in order to find a resolving set of $G \circ H$ we must find a subset $S_i(a) \subseteq H_i(a)$ for every $i \in \{1, 2, \ldots, k\}$ and $|V(H_i)| \geq 2$, such that every two distinct vertices $x, y \in H_i(a)$ satisfy $r(x|S_i(a)) \neq r(y|S_i(a))$, which can be seen in the following lemma.

Lemma 1. *Let* G *be a connected graph with* $|V(G)| \geq 2$ *and* H *be an arbitrary graph containing* $k \geq 1$ *components* $H_1, H_2, \ldots, H_k$ *and* $|V(H)| \geq 2$. *Let* W *be a basis of* $G \circ H$. *For any vertex* $a \in V(G)$, *if* $S_i(a) = W \cap H_i(a)$ *for every* $i \in \{1, 2, \ldots, k\}$ *where* $V(H_i) \geq 2$, *then* $S_i(a) \neq \emptyset$. *Moreover, if* B_i *is a basis of* H_i, *then* $|S_i(a)| \geq |B_i|$.

Proof. Suppose that there exists $a \in V(G)$ such that there exists $i \in \{1, 2, \ldots, k\}$ which is satisfying $|V(H_i)| \geq 2$ and $S_i(a) = \emptyset$. Since $|V(H_i)| \geq 2$, by Propositions 1 and 2, there exist two different vertices $(a, x), (a, y) \in H_i(a)$ such that $r((a, x)|W) = r((a, y)|W)$, a contradiction.

Now, suppose that $S_i(a) = \{(a, s_1), (a, s_2), \ldots, (a, s_t)\}$ where $t < |B_i|$ for some basis B_i of H_i. Let us consider $S' = \{s_1, s_2, \ldots, s_t\}$ subset of $V(H_i)$. Since $|S'| < |B_i|$, there exist two distinct vertices $x, y \in V(H_i)$ such that $r(x|S') = r(y|S')$. So, for every $p \in \{1, 2, \ldots, t\}$, we have $d(x, s_p) = d(y, s_p)$. Note that, for every two distinct vertices $u, v \in V(H_i)$, if $d(u, v) \leq 2$ then $d((a, u), (a, v)) = d(u, v)$, otherwise $d((a, u), (a, v)) = 2$. Thus we obtain $d((a, x), (a, s_p)) = d((a, y), (a, s_p))$, and so $r((a, x)|S_i(a)) = r((a, y)|S_i(a))$, a contradiction. □

For a graph H containing singleton components, we obtain the lemma below.

Lemma 2. *Let* G *be a connected graph with* $|V(G)| \geq 2$ *and* H *be an arbitrary graph containing* $k \geq 1$ *components* $H_1, H_2, \ldots, H_k$ *where* $1 \leq |V(H_1)| \leq |V(H_2)| \leq \cdots \leq |V(H_k)|$ *and* $|V(H)| \geq 2$. *Let* W *be a basis of* $G \circ H$. *For any vertex* $a \in V(G)$, *let* $W(a) = W \cap H(a)$. *If* H *contains* $m \geq 1$ *singleton components, then* $W(a)$ *contains at least* $m - 1$ *vertices of* $H_1(a) \cup H_2(a) \cup \cdots \cup H_m(a)$.

Proof. For $m = 1$, let $a \in V(G)$ and $x \in V(H_1)$. Let W be a resolving set of $H_2(a) \cup H_3(a) \cup \cdots \cup H_k(a)$. Note that, for a vertex $u \in V(H_i)$ and $v \in V(H) \setminus V(H_i)$ where $i \in \{1, 2, \ldots, k\}$, $d((a, u), (a, v)) = 2$. So, $r((a, x) \mid W) = (2, 2, \ldots, 2)$.

If every vertex $y \in V(H_j)$ with $j \in \{2, 3, \dots, k\}$ satisfies $r((a, y) \mid W) \neq (2, 2, \dots, 2)$ then we can choose $W(a) = W$ as a resolving set of $H(a)$. Otherwise, let $y \in V(H_2) \cup V(H_3) \cup \cdots \cup V(H_k)$ satisfies $r((a, y) \mid W) = (2, 2, \dots, 2)$. So, we can choose $W(a) = W \cup \{(a, y)\}$ as a resolving set of $H(a)$.

For $m \geq 2$, suppose that there exists $a \in V(G)$ such that $W(a)$ contains at most $m - 2$ vertices of $H_1(a) \cup H_2(a) \cup \cdots \cup H_m(a)$. Let x and y be two distinct vertices of $H_1(a) \cup H_2(a) \cup \cdots \cup H_m(a)$ which are not elements of $W(a)$. We obtain two situations.

1. For $z \in H(a) \setminus \{x, y\}$, we obtain $d(z, x) = 2 = d(z, y)$.
2. For any vertex $b \in V(G) \setminus \{a\}$, if $z \in H(b)$, then by Proposition 1, $d(z, x) = d(z, y)$.

From both situations, we obtain that $r(x \mid W) = r(y \mid W)$, a contradiction. □

Next, we consider $H_i(a)$ and $H(b)$ with $ab \in E(G)$ and $i \in \{1, 2, \dots, k\}$. By the definition of $G \circ H$, every vertex of $H(b)$ is adjacent to all vertices of $H_i(a)$. Now, we consider the induced subgraph from one vertex of $H(b)$ and all vertices of $H_i(a)$ which is isomorphic to a joint graph $H_i + K_1$. We will use a basis of $H_i + K_1$ to construct a resolving set of $G \circ H$. In order to do so, we show that we can always choose a basis of $H_i + K_1$ which is a subset of the vertex set of H_i. Note that H_i is a connected graph.

Lemma 3. *Let Q be a connected graph. There exists a basis S of $Q + K_1$ such that $S \subseteq V(Q)$.*

Proof. Let $V(Q + K_1) = V(Q) \cup \{v\}$ and S be a basis of $Q + K_1$. If $v \notin S$ we have nothing to prove. Suppose that $v \in S$. We distinguish two cases.

Case 1: $S \setminus \{v\} = \emptyset$.

By Theorem 1, $Q \cong K_1$. We obtain $Q + K_1 \cong P_2$. Chartrand et al. [3] and Khuller et al. [11] showed that $\beta(P_2) = 1$ where the vertex in a basis is one of P_2's end points. Since $Q + K_1$ has an end point which is a vertex of Q, we can choose a basis S' of $Q + K_1$ such that $v \notin S'$.

Case 2: $S \setminus \{v\} \neq \emptyset$.

We define $B = V(Q + K_1) \setminus S$. Let $r = |V(Q + K_1) \setminus S|$ and $B = \{b_1, b_2, \dots, b_r\}$. For $t \in \{1, 2, \dots, r\}$, we define a vertex set $S_t = (S \cup \{b_t\}) \setminus \{v\}$ and $B_t = B \setminus \{b_t\}$. If there exists $t \in \{1, 2, \dots, r\}$ such that every $u \in B_t$ satisfies $r(u|S_t) \neq (1, 1, \dots, 1)$, then the lemma is proved. Otherwise, we have that $Q + K_1$ is isomorphic to a complete graph. Chartrand et al. [3] has proved that the metric dimension of a complete graph K_n is $n - 1$. Then we can choose $S' = V(Q + K_1) \setminus \{v\}$ as a basis of $Q + K_1$. □

For $1 \leq i \leq k$, let B_i be a basis of $H_i + K_1$ such that $B_i \subseteq V(H_i)$. From Lemma 3, for $a \in V(G)$, choose a vertex set $W(a) = \bigcup_{1 \leq i \leq k} W_i$ where $W_i = \{(a, x)|x \in B_i\}$. In most cases, $W(a)$ resolves all vertices of $H(a)$. In Lemma 4, we give a condition for $W(a)$ which is not a resolving set of $H(a)$.

Lemma 4. *For $k \geq 1$ and $i \in \{1, 2, \dots, k\}$, let $a \in V(G)$, B_i be a basis of $H_i + K_1$ such that $B_i \subseteq V(H_i)$, and $W(a) = \bigcup_{1 \leq i \leq k} W_i(a)$ where $W_i(a) = \{(a, x)|x \in B_i\}$. For $x, y \in V(H)$, $r((a, x) \mid W(a)) = r((a, y) \mid W(a))$ if and only if $x \in V(H_i)$ and $y \in V(H_j)$ and $r(x \mid B_i) = (2, 2, \dots, 2)$ and $r(y \mid B_j) = (2, 2, \dots, 2)$ where $i \neq j$.*

Proof. (⇐) Since $r(x \mid B_i) = (2, 2, \dots, 2)$ and $r(y \mid B_j) = (2, 2, \dots, 2)$, we obtain that $r((a, x) \mid W_i(a)) = (2, 2, \dots, 2)$ and $r((a, y) \mid W_j(a)) = (2, 2, \dots, 2)$. Note that, for every $p, q \in \{1, 2, \dots, k\}$ and $p \neq q$, every vertices $u \in V(H_p(a))$ and $v \in V(H_q(a))$ satisfy $d(u, v) = 2$. Therefore, we obtain $r((a, x) \mid W(a)) = (2, 2, \dots, 2) = r((a, y) \mid W(a))$.

(⇒) For $1 \leq i \leq k$, since B_i is a basis of $H_i + K_1$, then every two distinct vertices $u, v \in H_i(a)$ satisfy $r(u \mid W_i(a)) \neq r(v \mid W_i(a))$. Therefore, since $r((a, x) \mid W(a)) = r((a, y) \mid W(a))$, we obtain that $(a, x) \in V(H_i(a))$ and $(a, y) \in V(H_j(a))$ where $i \neq j$, which implies $x \in V(H_i)$ and $y \in V(H_j)$.

Now, suppose that $r(x \mid B_i) \neq (2, 2, \dots, 2)$ or $r(y \mid B_j) \neq (2, 2, \dots, 2)$. So, there exists a vertex $u \in B_i$ or $v \in B_j$ such that $ux, yv \in E(H)$. Therefore, $d((a, u), (a, x)) = 1$ and $d((a, v), (a, y)) = 1$. Since $d((a, u), (a, y)) = 2$ and $d((a, v), (a, x)) = 2$, we obtain that $r((a, x) \mid W(a)) \neq r((a, y) \mid W(a))$, a contradiction. □

If the condition in Lemma 4 occurs then we must add more vertices on $W(a)$ such that the new set resolves (a, x) and (a, y).

Lemma 5. *Let G be a connected graph with $|V(G)| \geq 2$ and H be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, \dots, H_k$ and $|V(H)| \geq 2$. Let $a \in V(G)$ and W be a basis of $G \circ H$. If $W(a) = W \cap H(a)$ and $\alpha(a) = |W(a)|$, then*

$$\alpha(a) \leq \left(\sum_{p=1}^{k} \beta(H_p + K_1) \right) + k - 1.$$

Proof. For $i \in \{1, 2, \dots, k\}$, let B_i be a basis of $H_i + K_1$ such that $B_i \subseteq V(H_i)$. From Lemma 3, choose a vertex set $W_1 = \bigcup_{1 \leq i \leq k} W_i(a)$ where $W_i(a) = \{(a, x)|x \in B_i\}$. We distinguish two cases.

1. $G \circ H$ does not satisfy the conditions in Lemma 4.
   Then choose $W(a) = W_1$. Since B_i is a basis of $H_i + K_1$ for $1 \leq i \leq k$, then $W_i(a)$ resolves $V(H_i(a))$, which implies $W(a)$ resolves $V(H(a))$. Therefore, $\alpha(a) = \sum_{p=1}^{k} \beta(H_p + K_1) \leq \left(\sum_{p=1}^{k} \beta(H_p + K_1) \right) + k - 1$.

2. $G \circ H$ satisfies the conditions in Lemma 4.
   Let $S = \{(a, x) \mid (a, x) \in V(H_i(a)), r((a, x) \mid W_i(a)) = (2, 2, \ldots, 2), i \in \{1, 2, \ldots, k\}\}$. Note that $|S| \le k$. Let $z \in S$. We define $W_2 = S \setminus \{z\}$. Choose $W(a) = W_1 \cup W_2$. Since $W_i(a)$ resolves $V(H_i(a))$ for $1 \le i \le k$ and W_2 resolves S, then we obtain that $W(a)$ resolves $V(H(a))$. Therefore, $\alpha(a) \le \left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) + k - 1$. □

For $a \in V(G)$, let $W(a)$ be a resolving set of $H(a)$. By considering Proposition 1, choose $W = \bigcup_{a \in V(G)} W(a)$. In most cases, W is a resolving set of $G \circ H$. In Lemma 6, we give a condition for W which is not a resolving set of $G \circ H$.

We consider two different vertices $u, v \in V(G)$. Let $P_G(u, v)$ be a shortest $u - v$ path in G and $\ell(P_G(u, v))$ be the length of $P_G(u, v)$. We define $\mathscr{P}_G(u, v) = \{P_G(u, v)\}$. Let z be a vertex in $V(G) \setminus V(\mathscr{P}_G(u, v))$. If $\ell(P_G(u, v)) + \ell(P_G(v, z)) > \ell(P_G(u, z))$ and $\ell(P_G(u, v)) + \ell(P_G(u, z)) > \ell(P_G(v, z))$, then each path in $\mathscr{P}_G(u, v)$ is called an *eccentric path* of G.

Lemma 6. *Let $a, b \in V(G)$ and $W = \bigcup_{a \in V(G)} W(a)$ where $W(a)$ is a resolving set of $H(a)$. For $x, y \in V(H)$, $r((a, x) \mid W) = r((b, y) \mid W)$ if and only if $r((a, x) \mid W(a)) = (2, 2, \ldots, 2)$, $r((b, y) \mid W(b)) = (2, 2, \ldots, 2)$, and each shortest $a - b$ path is an eccentric path of length 2.*

Proof. (⇒) Suppose that $r((a, x) \mid W(a)) \ne (2, 2, \ldots, 2)$ or $r((b, y) \mid W(b)) \ne (2, 2, \ldots, 2)$ or each shortest $a - b$ path is an eccentric path of length $m \ne 2$. By using Lemma 3, there is no vertex z in H which has a representation $(1, 1, \ldots, 1)$ with respect to a basis of $H + K_1$. Now, we consider two cases.

1. $r((a, x) \mid W(a)) \ne (2, 2, \ldots, 2)$ or $r((b, y) \mid W(b)) \ne (2, 2, \ldots, 2)$.
   For either $ab \in E(G)$ and $ab \notin E(G)$, $r((a, x) \mid W(a)) \ne r((b, y) \mid W(a))$, which implies $r((a, x) \mid W) \ne r((b, y) \mid W)$, a contradiction.
2. Each shortest $a - b$ path is an eccentric path of length $m \ne 2$.
   If $m = 1$ then $r((a, x) \mid W(a)) \ne r((b, y) \mid W(a))$. Otherwise, there exists $c \in V(G)$ such that $bc \in E(G)$ and $ac \notin E(G)$, which implies $r((a, x) \mid W(c)) \ne r((b, y) \mid W(c))$. In both situations, we obtain $r((a, x) \mid W) \ne r((b, y) \mid W)$, a contradiction.

(⇐) Let $S = \{c \in V(G) \mid ac, cb \in E(G)\}$. For every $v \in V(G) \setminus (S \cup \{a, b\})$, we have $av, bv \notin E(G)$. Since G is a connected graph, for $u \in \{a, b\}$, there exists $c \in S$ such that the shortest $u - v$ path contains c. It follows $d(u, v) = d(u, c) + d(c, v)$. Note that, for $z \in V(H(v))$, $d((a, x), (v, z)) = d(u, v) = d((b, y), (v, z))$. Since $d(a, b) = 2$, we obtain $r((a, x) \mid W) = r((b, y) \mid W)$. □

If the condition in Lemma 6 occurs then we must add more vertices on W such that the new set resolves (a, x) and (b, y).

Lemma 7. *Let G be a connected graph with $|V(G)| \ge 2$ and H be an arbitrary graph containing $k \ge 1$ components $H_1, H_2, \ldots, H_k$ and $|V(H)| \ge 2$. If $|V(G)| = n$, then*

$$\beta(G \circ H) \le n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) + k - 1\right) + (n - 2).$$

Proof. For $a \in V(G)$, let $W(a)$ be a resolving set of $H(a)$. By considering Proposition 1, choose a vertex set $W_1 = \bigcup_{a \in V(G)} W(a)$. We distinguish two cases.

1. $G \circ H$ does not satisfy the condition in Lemma 6.
   Then choose $W = W_1$. Since $W(a)$ resolves $H(a)$ for every $a \in V(G)$, we obtain that W is a resolving set of $G \circ H$ and by Lemma 5,
$$\begin{aligned} |W| &\le n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) + k - 1\right) \\ &\le n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) + k - 1\right) + (n - 2). \end{aligned}$$
2. $G \circ H$ satisfies the condition in Lemma 6.
   From Lemma 6, we define $S_1 = \bigcup_{a \in V(G)} S(a)$ where $S(a) = \{b \in V(G) \mid$ each shortest $a - b$ path is an eccentric path of length 2$\}$. Note that $|S_1| \le n - 1$. Let $S_2 = \{(b, x) \mid b \in S_1$ and $r((b, x) \mid W(b)) = (2, 2, \ldots, 2)\}$. Let $z \in S_2$. We define $W_2 = S_2 \setminus \{z\}$. Then choose a vertex set $W = W_1 \cup W_2$. Since $W(a)$ resolves $H(a)$ for every $a \in V(G)$ and W_2 resolves S_2, we obtain that W is a resolving set of $G \circ H$ and by Lemma 5, $|W| \le n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) + k - 1\right) + (n - 2)$. □

Combining the results in Lemmas 1, 2 and 7, we obtain the following bounds of $\beta(G \circ H)$.

Theorem 2. *Let G be a connected graph with $|V(G)| \geq 2$ and H be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, \ldots, H_k$ and $|V(H)| \geq 2$. If $|V(G)| = n$, then*

$$n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p) \right) - 1 \right) \leq \beta(G \circ H) \leq n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1) \right) + k - 1 \right) + (n-2).$$

In the next two subsections, we prove that the lower and upper bounds in Theorem 2 are sharp.

2.1. H is a disconnected graph

In the next two theorems, we prove the existence of a connected graph G and a disconnected graph H where the metric dimension of $G \circ H$ satisfies either the lower or upper bounds in Theorem 2.

Theorem 3. *There exists a connected graph G of order $n \geq 2$ and a graph H containing $k \geq 2$ components $H_1, H_2, \ldots, H_k$ such that*

$$\beta(G \circ H) = n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p) \right) - 1 \right).$$

Proof. Let G be a path of n vertices P_n where $n \geq 4$, and H be a null graph (graph without edges) of k vertices where $k \geq 2$. By Theorem 2, we only need to show that $\beta(G \circ H) \leq n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p) \right) - 1 \right)$.

Let $V(G) = \{p_1, p_2, \ldots, p_n\}$ where $p_i p_{i+1} \in E(G)$ for $1 \leq i \leq n-1$, and $V(H) = \{q_1, q_2, \ldots, q_k\}$. We define $W = V(G \circ H) \setminus G(q_k)$. We will show that W is a resolving set of $G \circ H$. Note that,

1. $d((p_i, q_k), (p_{j+1}, q_1)) \neq d((p_j, q_k), (p_{j+1}, q_1))$ for $1 \leq i \leq j \leq n-1$.
2. $d((p_n, q_k), (p_{i-1}, q_1)) \neq d((p_i, q_k), (p_{i-1}, q_1))$ for $2 \leq i \leq n-1$.
3. $d((p_1, q_k), (p_2, q_1)) \neq d((p_n, q_k), (p_2, q_1))$.

Therefore, since $r(u \mid W) \neq r(v \mid W)$ for every two distinct vertices $u, v \in V(G \circ H)$, W is a resolving set of $G \circ H$. □

Theorem 4. *There exists a connected graph G of order $n \geq 2$ and a graph H containing $k \geq 2$ components $H_1, H_2, \ldots, H_k$ such that*

$$\beta(G \circ H) = n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1) \right) + k - 1 \right) + (n-2).$$

Proof. Let G be a star of n vertices S_{n-1} where $n \geq 4$, and H be a graph containing $k \geq 2$ components $H_1, H_2, \ldots, H_k$ where H_i is a path of 8 vertices P_8 for $1 \leq i \leq k$. By Theorem 2, we only need to show that $\beta(G \circ H) \geq n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1) \right) + k - 1 \right) + (n-2)$. Note that $P_8 + K_1$ is a fan graph with 9 vertices. Caceres et al. [2] have proved that $\beta(P_8 + K_1) = \left\lfloor \frac{2 \cdot 8 + 2}{5} \right\rfloor = 3$. If B is a basis of $P_8 + K_1$, then there exists a vertex $y \in V(P_8)$ such that $r(y \mid B) = (2, 2, 2)$.

Suppose that $\beta(G \circ H) \leq n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1) \right) + k - 1 \right) + (n-3)$. Let W be a basis of $G \circ H$. Then there exist two distinct leaves $a, b \in V(G)$ such that each leaf contributes at most $\left(\sum_{p=1}^{k} \beta(H_p + K_1) \right) + k - 1$ vertices in W. Let c be the center vertex of G. Note that, for every vertex $z \in V(G) \setminus \{a, b, c\}$, $d(z, u) = d(z, c) + d(c, u)$ for $u \in \{a, b\}$. Since $P_G(a, b)$ is an eccentric path of length 2 and $d((u, y), (z, w)) = d(u, z)$ for $w \in V(H)$, we obtain that $r((a, y) \mid W) = r((b, y) \mid W)$, a contradiction. □

The graph in the proof of Theorem 4 satisfies the condition in Lemmas 4 and 6. In particular, in Theorems 5 and 6 we give an example of graphs with metric dimension $n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1) \right) + k - 1 \right)$ and $n \cdot \left(\sum_{p=1}^{k} \beta(H_p + K_1) \right)$, respectively.

Theorem 5. *There exists a connected graph G of order $n \geq 2$ and a graph H containing $k \geq 2$ components $H_1, H_2, \ldots, H_k$ such that*

$$\beta(G \circ H) = n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1) \right) + k - 1 \right).$$

Proof. Let G be a complete graph of n vertices K_n and H be a graph containing $k \geq 2$ components $H_1, H_2, \ldots, H_k$ where H_i is a cycle of 8 vertices C_8 for $1 \leq i \leq k$. Note that $G \circ H$ does not satisfy the condition in Lemma 6. We need to show that $\beta(G \circ H) = n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1) \right) + k - 1 \right)$. The $C_8 + K_1$ is a wheel graph with 9 vertices. Buczkowski et al. [1] have proved that $\beta(C_8 + K_1) = 3$. If B is a basis of $C_8 + K_1$, then there exists a vertex $y \in V(C_8)$ such that $r(y \mid B) = (2, 2, 2)$.

For $1 \le i \le k$, let B_i be a basis of $H_i + K_1$ where $B_i \subseteq V(H_i)$. For $a \in V(G)$, let $S_1(a) = \{(a, y) \mid y \in V(H_i); r(y \mid B_i) = (2, 2, 2); 1 \le i \le k\}$. Let $W_i(a) = \{(a, x) \mid x \in B_i\}$ and for a vertex $z \in S_1(a)$, let $S_2(a) = S_1(a) \setminus \{z\}$. We define $W(a) = S_2(a) \cup \bigcup_{1 \le i \le k} W_i(a)$. Since $W_i(a)$ resolves $H_i(a)$ and $S_2(a)$ resolves $S_1(a)$, we obtain that $W(a)$ is a resolving set of $H(a)$ and $|W(a)| = \left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) + k - 1$. By considering Proposition 1, $W = \bigcup_{a \in V(G)} W(a)$ is a resolving set of $G \circ H$ and $|W| = n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) + k - 1\right)$.

Now, suppose that $\beta(G \circ H) \le n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) + k - 1\right) - 1$. Let W be a basis of $G \circ H$. Then there exists a vertex $a \in V(G)$ such that $H(a)$ contributes at most $\left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) + k - 2$ vertices in W. For $1 \le i \le k$, let $W_i(a) = W \cap H_i(a)$. If there exists $i \in \{1, 2, \dots, k\}$ such that $|W_i(a)| < \beta(C_8 + K_1)$, then there exist two vertices $x, y \in H_i(a)$ such that $r(x \mid W_i(a)) = r(y \mid W_i(a))$, which implies $r(x \mid W) = r(y \mid W)$, a contradiction. So, for every $i \in \{1, 2, \dots, k\}$, we assume that $|W_i(a)| \ge \beta(C_8 + K_1)$. Therefore, there exist two different components H_i and H_j such that $|W_i(a)| = \beta(C_8 + K_1) = |W_j(a)|$. Let $x \in V(H_i)$ and $y \in V(H_j)$ where $r((a, x) \mid W_i(a)) = (2, 2, 2) = r((a, y) \mid W_j(a))$. By Lemma 4, we obtain that $r((a, x) \mid W) = r((a, y) \mid W)$, a contradiction. □

Theorem 6. *There exists a connected graph G of order $n \ge 2$ and a graph H containing $k \ge 2$ components $H_1, H_2, \dots, H_k$ such that*

$$\beta(G \circ H) = n \cdot \left(\sum_{p=1}^{k} \beta(H_p + K_1)\right).$$

Proof. Let G be a complete graph of n vertices K_n and H be a graph containing $k \ge 2$ components $H_1, H_2, \dots, H_k$ where H_i is a path of 4 vertices P_4. Note that $G \circ H$ does not satisfy the conditions in Lemmas 4 and 6. The $P_4 + K_1$ is a fan graph with 5 vertices. Caceres et al. [2] have proved that $\beta(P_4 + K_1) = 2$.

For $1 \le i \le k$, let B_i be a basis of $H_i + K_1$ where $B_i \subseteq V(H_i)$. For $a \in V(G)$, let $W_i(a) = \{(a, x) \mid x \in B_i\}$. We define $W(a) = \bigcup_{1 \le i \le k} W_i(a)$. Since $W_i(a)$ resolves $H_i(a)$, we obtain that $W(a)$ is a resolving set of $H(a)$ and $|W(a)| = \sum_{p=1}^{k} \beta(H_p + K_1)$. By considering Proposition 1, $W = \bigcup_{a \in V(G)} W(a)$ is a resolving set of $G \circ H$ and $|W| = n \cdot \left(\sum_{p=1}^{k} \beta(H_p + K_1)\right)$.

Now, suppose that $\beta(G \circ H) \le n \cdot \left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) - 1$. Let S be a basis of $G \circ H$. Then there exists a vertex $a \in V(G)$ such that $H(a)$ contributes at most $\left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) - 1$ vertices in S. For $1 \le i \le k$, let $S_i(a) = S \cap H_i(a)$. Then there exists $i \in \{1, 2, \dots, k\}$ such that $|S_i(a)| < \beta(P_4 + K_1)$. Therefore, there exist two vertices $x, y \in H_i(a)$ such that $r(x \mid W_i(a)) = r(y \mid W_i(a))$, which implies $r(x \mid W) = r(y \mid W)$, a contradiction. □

An interesting question is whether all the values between the lower and the upper bounds are achievable, as stated in the following problem.

Problem 1. Let H be a graph containing $k \ge 2$ components $H_1, H_2, \dots, H_k$. For every integer c with

$$n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p)\right) - 1\right) < c < n \cdot \left(\left(\sum_{p=1}^{k} \beta(H_p + K_1)\right) + k - 1\right) + (n - 2),$$

does there exist a connected graph G of order n such that $\beta(G \circ H) = c$?

2.2. H is a connected graph

For H is a connected graph with $|V(H)| \ge 2$, then H is not a singleton component and $k = 1$. So, $G \circ H$ does not satisfy a condition in Lemma 2. Therefore, combining the results in Lemmas 1 and 7, we obtain the following bounds of $\beta(G \circ H)$.

Theorem 7. *Let G and H be connected graphs with $|V(G)| \ge 2$ and $|V(H)| \ge 2$. If $|V(G)| = n$, then*

$$n \cdot \beta(H) \le \beta(G \circ H) \le n \cdot \beta(H + K_1) + (n - 2).$$

In the next two theorems, we prove the existence of connected graphs G and H where the metric dimension of $G \circ H$ satisfies either the lower or upper bounds in Theorem 7.

Theorem 8. *There exist connected graphs G of order $n \ge 2$ and H of order at least 2 such that $\beta(G \circ H) = n \cdot \beta(H)$.*

Proof. Let G be an arbitrary connected graph and H be a graph with diameter at most 2. Generally, for a graph H with diameter at most 2, the metric dimension of $G \circ H$ is equal to the lower bound of Theorem 7 since two distinct vertices $x, y \in V(H)$ and a vertex $a \in V(G)$ satisfy $d(x, y) = d((a, x), (a, y))$. Therefore, for every $a \in V(G)$, $H(a)$ contributes at least $\beta(H)$ vertices in a basis of $G \circ H$. □

Theorem 9. *There exist connected graphs G of order $n \geq 2$ and H of order at least 2 such that* $\beta(G \circ H) = n \cdot \beta(H + K_1) + (n-2)$.

Proof. Let $H \cong P_8$ be a path with 8 vertices and $G \cong S_{n-1}$ be a star with n vertices with $n \geq 3$. By Theorem 7, we only need to show that $\beta(G \circ H) \geq n \cdot \beta(H + K_1) + (n-2)$. The $H + K_1$ graph is a fan graph with 9 vertices. Caceres et al. [2] have proved that $\beta(P_8 + K_1) = \left\lfloor \frac{2 \cdot 8 + 2}{5} \right\rfloor = 3$. If B is a basis of $H + K_1$, then there exists a vertex $y \in V(H)$ such that $r(y \mid B) = (2, 2, 2)$.

Suppose that $\beta(G \circ H) \leq n \cdot \beta(H + K_1) + (n-3)$. Let W be a basis of $G \circ H$. Then there exist two distinct leaves $a, b \in V(G)$ such that each leaf contributes at most $\beta(H + K_1)$ vertices in W. Let $c \in V(G)$ be the center vertex of G. Note that every vertex $z \in V(G) \setminus \{a, b, c\}$ satisfies $d(z, u) = d(z, c) + d(c, u)$ for $u \in \{a, b\}$. Since $a - b$ path is an eccentric path of length 2 and $d((z, w), (u, y)) = d(z, u)$ for $w \in V(H)$, we obtain that $r((a, y) \mid W) = r((b, y) \mid W)$, a contradiction. □

Note that the graph in the proof of Theorem 9 satisfies the condition in Lemma 6. In particular, in Theorem 10 we have an example of graphs $G \circ H$ with metric dimension $n \cdot \beta(H + K_1)$.

Theorem 10. *There exist connected graphs G of order $n \geq 2$ and H of order at least 2 such that* $\beta(G \circ H) = n \cdot \beta(H + K_1)$.

Proof. Let $H \cong P_m$ be a path with $m \geq 7$ vertices and $G \cong P_n$ be a path with $n \geq 4$ vertices. Note that $G \circ H$ does not satisfy the condition in Lemma 6. We need to show that $\beta(G \circ H) = n \cdot \beta(H + K_1)$. The $H + K_1$ graph is a fan graph with $m + 1$ vertices. Caceres et al. [2] have proved that $\beta(P_m + K_1) = \left\lfloor \frac{2m+2}{5} \right\rfloor$.

Let B be a basis of $H + K_1$. From Lemma 3, choose a vertex set $W = \bigcup_{v \in V(G)} W(v)$ where $W(v) = \{(v, x) \mid x \in B\}$. Since for every $v \in V(G)$, $W(v)$ resolves $H(v)$, then by considering Proposition 1, W is a resolving set of $G \circ H$. Therefore, $\beta(G \circ H) \leq n \cdot \beta(H + K_1)$.

Now, suppose that $\beta(G \circ H) \leq n \cdot \beta(H + K_1) - 1$. Let S be a basis of $G \circ H$. Then there exists a vertex $a \in V(G)$ such that $H(a)$ contributes $\beta(H + K_1) - 1$ vertices in S. So, there exist two vertices $x, y \in V(H)$ such that $r((a, x) \mid S) = r((a, y) \mid S)$, a contradiction. Therefore, $\beta(G \circ H) \geq n \cdot \beta(H + K_1)$. □

We can also show that there exist graphs G and H such that the metric dimension of $G \circ H$ is not equal to both the lower and upper bounds in Theorem 7.

Theorem 11. *There exist connected graphs G of order $n \geq 2$ and H of order at least 2 such that $\beta(G \circ H) = c$ where* $n \cdot \beta(H) < c < n \cdot \beta(H + K_1) + (n-2)$.

Proof. Let $G \cong K_n$ be a complete graph with $n \geq 2$ vertices and $H \cong K_m$ be a complete graph with $m \geq 2$ vertices. Since $G \circ H \cong K_{mn}$, we have $\beta(G \circ H) = mn - 1$ (see [3]). Since $H + K_1 \cong K_{m+1}$, we obtain $n \cdot \beta(H) < nm - 1 < n \cdot \beta(H + K_1) + (n-2)$. □

An interesting question is whether all the values between the lower and the upper bounds are achievable, as stated in the following problem.

Problem 2. Let H be a connected graph of order at least 2. For every integer c with $n \cdot \beta(H) < c < n \cdot \beta(H + K_1) + (n-2)$, does there exist a graph G of order n such that $\beta(G \circ H) = c$?

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