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Note

Large bipartite Cayley graphs of given degree and diameter

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ABSTRACT

Let $BC_{d,k}$ be the largest possible number of vertices in a bipartite Cayley graph of degree d and diameter k. We show that $BC_{d,k} \geq 2(k-1)((d-4)/3)^{k-1}$ for any $d \geq 6$ and any even $k \geq 4$, and $BC_{d,k} \geq (k-1)((d-2)/3)^{k-1}$ for $d \geq 6$ and $k \geq 7$ such that $k \equiv 3 \pmod 4$.

The problem of determining the largest graphs of given maximum degree and diameter is known as the degree–diameter problem. For a detailed survey on the problem we refer the reader to [5]. In this note we focus on bipartite Cayley graphs of given degree and diameter. Let $B_{d,k}$ denote the largest order of a bipartite graph of maximum degree d and diameter k, and let $BC_{d,k}$ be the largest number of vertices in a bipartite Cayley graph of degree d and diameter k.

Biggs [2] showed that the number of vertices in a bipartite graph of degree d and diameter k cannot exceed the bipartite Moore bound $M_{d,k} = 2((d-1)^k - 1)/(d-2)$ for $d \geq 3$, and the bound $2k$ for $d = 2$. Exact values of $B_{d,k}$ and $BC_{d,k}$ are available only in rare cases. For example, it is easy to see that for $d, k \geq 2$, we have $B_{2,k} = 2k$ and $B_{d,2} = 2d$. The graphs of order $2k$ are the $2k$-cycles and the graphs of order $2d$ are the complete bipartite graphs $K_{d,d}$. Since both cycles and complete bipartite graphs with partite sets of equal size are Cayley graphs, $BC_{2,k} = 2k$ and $BC_{d,2} = 2d$ as well. It is also known that $B_{d,k}$ is equal to the bipartite Moore bound if $k = 3, 4, 6$ and $d-1$ is a prime power; see [1,2].

Improvements on the upper bound for $B_{d,k}$ can be seen in the recent papers of Pineda-Villavicencio [6] and Delorme et al. [4]. Pineda-Villavicencio [6] proved that there exist no bipartite graphs of order $M_{d,k} - 2$ for any $d \geq 3$ and $k \geq 4$, which yields the bound $B_{d,k} \leq M_{d,k} - 4$ for any $d \geq 3$ and $k \geq 5$ with $k \neq 6$. There are no general upper bounds on $BC_{d,k}$ better than the upper bounds for $B_{d,k}$.

For the largest known bipartite graphs of degree d and diameter k for small d and k, see [8]. The orders of known constructions of bipartite graphs for large d and k are significantly lower than the bipartite Moore bound. Bond and Delorme [3] presented large bipartite graphs of given degree and diameter using their concept of a partial Cayley graph. By suppressing edge directions in constructions of directed graphs of Vetrík [7], one has the general lower bound $BC_{d,k} \geq (k-1)((d-1)/4)^{k-1}$ for any $k \geq 4$ and any even $d \geq 8$.

We give bipartite Cayley graphs larger than bipartite graphs coming from constructions of [7]. First, let us present a family of bipartite Cayley graphs of degree $d \equiv 0 \pmod 3$.

Theorem 1. *Let $d \geq 6$ be a multiple of 3.*

(i) *For any even $k \geq 4$, we have $BC_{d,k} \geq 2(k-1)(d/3)^{k-1}$.*
(ii) *For $k \geq 7$ such that $k \equiv 3 \pmod 4$, we have $BC_{d,k} \geq (k-1)(d/3)^{k-1}$.*

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Proof. Let H be a group of order $m \geq 2$ with unit element ι and let H^{k-1} be the product $H \times H \times \cdots \times H$, where H appears $k-1$ times. Denote by α the automorphism of the group H^{k-1} such that $\alpha(x_1, x_2, \ldots, x_{k-1}) = (x_{k-1}, x_1, x_2, \ldots, x_{k-2})$. This means that α shifts coordinates by one to the right. The cyclic group of order t will be denoted by Z_t and the semidirect product $H^{k-1} \rtimes Z_t$ will be denoted by G. (Note that G is the semidirect product of N and A, written as $G = N \rtimes A$, if N is a normal subgroup of G, A is a subgroup of G, and every element of G can be written as a unique product of an element of N and an element of A.) Multiplication in G is given by $(x, y)(x', y') = (x\alpha^y(x'), y + y')$, where $x, x' \in H^{k-1}$, $y, y' \in Z_t$ and α^y is the composition of α with itself y times. We will write elements of G in the form $(x_1, x_2, \ldots, x_{k-1}; y)$, where $x_1, x_2, \ldots, x_{k-1} \in H$ and $y \in Z_t$.

(i) Let $k \geq 4$ be even and $t = 2(k-1)$. Let $a_g = (g, \iota, \ldots, \iota; 1)$ for any $g \in H$. Then $a_g^{-1} = (\iota, \ldots, \iota, g^{-1}; -1)$. Further, for any $h \in H$ let $b_h = (\iota, \ldots, \iota, h, \iota, \ldots, \iota; k-1)$, where $x_{k/2} = h$ and $x_j = \iota$ for $1 \leq j \leq k-1, j \neq k/2$.

Let $X = \{a_g, a_g^{-1}, b_h$ for all $g, h \in H\}$. Since $b_h^{-1} = b_{h^{-1}}$, it is clear that $X = X^{-1}$. The Cayley graph $C(G, X)$ is of degree $d = |X| = 3m$ and order $|G| = 2(k-1)m^{k-1} = 2(k-1)(d/3)^{k-1}$. We prove that the diameter of $C(G, X)$ is $\leq k$, which is equivalent to showing that each element of G can be expressed as a product of at most k elements of X.

We show that any element $(x_1, x_2, \ldots, x_{k-1}; y)$, where $x_1, x_2, \ldots, x_{k-1} \in H$ and $y \in Z_{2(k-1)}$, y is odd, can be obtained as a product of $k-1$ elements of X. Let $0 \leq r \leq k/2 - 1$. Let

$$S = (\Pi_{i=1}^{k/2-r} a_{x_i})(\Pi_{j=1}^{r} b_{x_{j+k-r-1}} a_{x_{j+k/2-r}})(\Pi_{\ell=k/2+1}^{k-r-1} a_{x_\ell})$$

and

$$S' = (\Pi_{i=1}^{k/2-r} a_{x_i}^{-1})(\Pi_{j=1}^{r} b_{x_{j+k-r-1}}^{-1} a_{x_{j+k/2-r}}^{-1})(\Pi_{\ell=k/2+1}^{k-r-1} a_{x_\ell}^{-1}).$$

It can be checked that $S = (x_1, x_2, \ldots, x_{k-1}; 2k-2-r)$ if r is odd, and $S = (x_1, x_2, \ldots, x_{k-1}; k-1-r)$ if r is even. Then $S' = (x_{k-1}^{-1}, x_{k-2}^{-1}, \ldots, x_1^{-1}; r)$ if r is odd, and $S' = (x_{k-1}^{-1}, x_{k-2}^{-1}, \ldots, x_1^{-1}; k-1+r)$ if r is even.

Any element $(x_1, x_2, \ldots, x_{k-1}; y+1)$ can be expressed as follows:

$$(x_1, x_2, \ldots, x_{k-1}; y+1) = (x_1, x_2, \ldots, x_{k-1}; y)a_\iota.$$

None of the elements $(x_1, x_2, \ldots, x_{k-1}; y)$ where all $x_i \neq \iota$, $1 \leq i \leq k-1$, can be obtained as a product of fewer than $k-1$ elements of X; therefore none of the elements $(x_1, x_2, \ldots, x_{k-1}; y+1)$ can be obtained as a product of fewer than k elements of X. The diameter of $C(G, X)$ is exactly k.

Since the last coordinate of any element in the generating set X is odd, no two different vertices $(x_1, x_2, \ldots, x_{k-1}; y)$ and $(x'_1, x'_2, \ldots, x'_{k-1}; y')$ of $C(G, X)$ are adjacent if either both y, y' are even or both y, y' are odd ($y, y' \in Z_{2(k-1)}$ and $x_i, x'_i \in H$, $1 \leq i \leq k-1$). It follows that the graph $C(G, X)$ is bipartite.

(ii) Let $k \geq 7$ such that $k \equiv 3 \pmod 4$ and $t = k-1$. The semidirect product $H^{k-1} \rtimes Z_{k-1}$ will be denoted by G'. For any $g, h \in H$ let $a_g = (g, \iota, \ldots, \iota; 1)$ and $b_h = (\iota, \ldots, \iota, h, \iota, \ldots, \iota, h; (k-1)/2)$, where $x_s = h$ for $s = (k-1)/2$ and $s = k-1$. It is evident that $b_h^{-1} = b_{h^{-1}}$.

Let $X' = \{a_g, a_g^{-1}, b_h;\ g, h \in H\}$. The Cayley graph $C(G', X')$ is of degree $d = |X'| = 3m$ and order $|G'| = (k-1)m^{k-1} = (k-1)(d/3)^{k-1}$. In order to prove that the diameter of $C(G', X')$ is equal to k it is enough to show that any element $(x_1, x_2, \ldots, x_{k-1}; y)$, where $x_i \in H$, $1 \leq i \leq k-1$ and $y \in Z_{k-1}$, y is even, can be expressed as a product of $k-1$ elements of X'. If $r = 0, 2, \ldots, (k-3)/2$, we have

$$(x_1, x_2, \ldots, x_{k-1}; k-1-r) = (\Pi_{i=1}^{(k-1)/2-r} a_{x_i})(\Pi_{j=1}^{r/2} a_{x_{2j+(k-3)/2-r} x_{2j+k-r-2}^{-1}} b_{x_{2j+k-r-2}} a_{x_{2j+k-r-1} x_{2j+(k-1)/2-r}^{-1}} b_{x_{2j+(k-1)/2-r}}) \times (\Pi_{\ell=(k+1)/2}^{k-r-1} a_{x_\ell}).$$

Elements of G' with the last coordinate r, where $r \in \{2, 4, \ldots, (k-3)/2\}$, can be obtained as inverses of the above ones. The diameter of $C(G', X')$ is k and it can be easily checked that $C(G', X')$ is bipartite. □

We modify the generating set given in the previous proof to get lower bounds on $BC_{d,k}$ for any $d \geq 6$.

Theorem 2. *Let $d \geq 6$ be an integer.*

(i) *If $k \geq 4$ is even, then $BC_{d,k} \geq 2(k-1)((d-4)/3)^{k-1}$.*
(ii) *If $k \geq 7$ such that $k \equiv 3 \pmod 4$, then $BC_{d,k} \geq (k-1)((d-2)/3)^{k-1}$.*

Proof. We use the notation of the proof of Theorem 1.

(i) By Theorem 1, $BC_{d,k} \geq 2(k-1)(d/3)^{k-1}$ for $d = 3m$, $m \geq 2$ and for any even $k \geq 4$. Let u, v be two different elements of G with an odd last coordinate such that $u, v \notin X$, $u \neq u^{-1}$ and $v \neq v^{-1}$. It is clear that such elements exist. Let $X_1 = X \cup \{u, u^{-1}\}$ and $X_2 = X \cup \{u, u^{-1}, v, v^{-1}\}$. Then, the Cayley graph $C(G, X_1)$ is bipartite, $C(G, X_1)$ has degree $d = |X_1| = 3m + 2$, diameter at most k and order $|G| = 2(k-1)m^{k-1} = 2(k-1)((d-2)/3)^{k-1}$. The bipartite Cayley graph $C(G, X_2)$ is of degree $d = |X_2| = 3m + 4$ and order $2(k-1)((d-4)/3)^{k-1}$.

Moreover, if m is even, the group G must contain an involution other than the identity, say z, not appearing in X. Let $X_3 = X \cup \{z\}$. The Cayley graph $C(G, X_3)$ is bipartite of degree $d = |X_3| = 3m + 1$, diameter at most k and order $|G| = 2(k-1)((d-1)/3)^{k-1}$. Thus, $BC_{d,k} \geq 2(k-1)((d-4)/3)^{k-1}$ for any $d \geq 6$ and any even $k \geq 4$.

(ii) From Theorem 1 it follows that $BC_{d,k} \geq (k-1)(d/3)^{k-1}$ if $d \geq 6$ is a multiple of 3 and $k \equiv 3 (\mathrm{mod}\ 4)$, $k \geq 7$. Let z, u be elements of G' with an odd last coordinate such that $z, u \notin X'$, the order of z is 2 and the order of u is greater than 2. Note that z must be of the form $(x_1, x_2, \ldots, x_{(k-1)/2}, x_1^{-1}, x_2^{-1}, \ldots, x_{(k-1)/2}^{-1}, (k-1)/2)$, where $x_1, x_2, \ldots, x_{(k-1)/2}$ are elements of H.

Let $X_1' = X' \cup \{z\}$ and $X_2' = X' \cup \{u, u^{-1}\}$. Then, the Cayley graph $C(G', X_1')$ is bipartite, and $C(G', X_1')$ has degree $d = |X_1'| = 3m + 1$, diameter at most k and order $|G'| = (k-1)m^{k-1} = 2(k-1)((d-1)/3)^{k-1}$. The bipartite Cayley graph $C(G', X_2')$ is of degree $d = 3m + 2$ and order $(k-1)((d-2)/3)^{k-1}$. $BC_{d,k} \geq (k-1)((d-2)/3)^{k-1}$ for any $d \geq 6$ and any $k \geq 7$ such that $k \equiv 3 (\mathrm{mod}\ 4)$. □

To the best of our knowledge there is no construction of bipartite graphs of order greater than the order of our graphs. Hence, for sufficiently large d and k, our graphs appear to be the largest known bipartite Cayley graphs of degree $d \geq 6$ and diameter $k \geq 4$, where $k \not\equiv 1 (\mathrm{mod}\ 4)$.

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