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THE LOCATING-CHROMATIC NUMBER OF DISCONNECTED GRAPHS

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Abstract

The paper generalizes the notion of locating-chromatic number of a graph such that it can be applied to disconnected graphs as well. In this sense, not all the graphs will have finite locating-chromatic numbers. We derive conditions under which a graph has a finite locating-chromatic number. In particular, we determine the locating-chromatic number of a uniform linear forest, namely a disjoint union of some paths with the same length.

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1. Introduction

The concept of the locating-chromatic number of graphs introduced by Chartrand et al. [10] is only applied for connected graphs. Many important results have been obtained. For instances, Chartrand et al. [10] determined the locating-chromatic number for cycles and complete multipartite graphs. Furthermore, Chartrand et al. [11] characterized all graphs of order n with the locating-chromatic number $n - 1$. In the same paper, Chartrand et al. [11] also gave some conditions for graphs of n vertices under which $n - 2$ is an upper bound of its locating-chromatic number. Recently, Asmiati and Baskoro [1] characterized all graphs on n vertices containing cycles with locating-chromatic number 3.

Some authors also studied the locating-chromatic number for graphs produced by some graph operations. Baskoro and Purwasih [9] determined the locating-chromatic number for the corona product of two graphs. Behtoei and Ommoomi determined the locating-chromatic number for Kneser graph [5], Cartesian product of graph [6] and join of graph [7].

The locating-chromatic number for trees was firstly studied by Chartrand et al. in 2002, by showing such a number for paths and double stars. Furthermore, Chartrand et al. [11] also showed that for any integer $k \in [3, n]$ and $k \neq n - 1$, there exists a tree on n vertices with the locating-chromatic number k. Asmiati et al. determined the locating-chromatic number of firecrackers [2] and amalgamation of stars [3]. Welyyanti et al. [8] studied the locating-chromatic number of complete n-ary trees. Recently, Baskoro and Asmiati [4] completed the characterization of all graphs with locating-chromatic number 3 by showing all trees having this number.

In this paper, we extend the definition of locating-chromatic number such that this concept can be applied to all graphs, including disconnected ones. Let c be a k-coloring on a disconnected graph $H(V, E)$. Let $\prod =$ $\{C_1, C_2, ..., C_k\}$ be the partition of $V(H)$ induced by c, where C_i is the set of all vertices receiving color i. The *color code* $c_{\Pi}(v)$ of a vertex $v \in H$

is the ordered k-tuple $(d(v, C_1), d(v, C_2), \ldots, d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x) | x \in C_i\}$ and $d(v, C_i) < \infty$ for all $i \in [1, k]$. If all vertices of H have distinct color codes, then c is called a *locating k-coloring* of H. The *locating-chromatic number* of H, denoted by $\chi'_L(H)$, is the smallest k such that H admits a locating-coloring with k colors. If there is no integer k satisfying the above conditions, then we say that $\chi'_L(H) = \infty$. Note that the locating-chromatic number of a connected graph G is denoted by $\chi_L(G)$.

2. Main Results

The following theorem gives the bounds of the locating-chromatic number of a disconnected graph if it is finite.

Theorem 2.1. *For each i, let G_i be a connected graph and let $H = \bigcup_{i=1}^{m} G_i$. If $\chi'_L(H) < \infty$, then $q \leq \chi'_L(H) \leq r$, where $q = \max\{\chi_L(G_i) : i \in [1, m]\}$ and $r = \min\{|V(G_i)| : i \in [1, m]\}$.*

Proof. Since $q = \max\{\chi_L(G_i) | i \in [1, m]\}$, there is an integer $k \in [1, m]$ such that $\chi_L(G_k) = q$. It means that every locating-coloring of graph H must have at least q colors in every component of H. Therefore, $\chi'_L(H) \geq q$. Next, we will show the upper bound of $\chi'_L(H)$. Since $r = \min\{| G_i || i \in [1, m]\}$, there is an integer $k \in [1, m]$ such that $\chi_L(G_k) = r$. It means that every locating-coloring of H must have at most r colors in every component of H. Therefore, $\chi'_L(H) \leq r$. □

For any locating-coloring c of graph H, define a *dominant vertex* as a vertex with $d(v, S) = 1$ if v is not in the color set S under c. The following theorem shows the locating-chromatic number of a disjoint union of s copies of a connected graph G, provided G has exactly one dominant vertex in its every locating-coloring.

Theorem 2.2. *Let G be a connected graph with $\chi_L(G) = k$ and $H = sG$. Let G has exactly one dominant vertex in its every locating-coloring. Then $\chi'_L(H) = k$ if $s \leq k$, otherwise $\chi'_L(H) = \infty$.*

Proof. If $s \leq k$, then $\chi'_L(G) \geq k$ by Theorem 2.1. ($\Rightarrow$) Since $\chi'_L(H) < \infty$, $\chi'_L(H) = k$. Let c_G be a locating k-coloring of G. Now, define $c_H(x) = x + i \bmod k$ if x is in the component G_i of H, $i \in [1, s]$. It can be verified that c_H is a locating-coloring of H, if $s \leq k$. Now, let $s > k$. Since any locating-coloring of H is also a locating-coloring of G, there are s dominant vertices, a contradiction. Thus, $\chi'_L(H) = \infty$. □

Next, we will determine the locating-chromatic number of a galaxy $H = \bigcup_{i=1}^{t} K_{1, n_i}$, where K_{1, n_i} is a star for $i \in [1, t]$.

Theorem 2.3. *Let $H = \bigcup_{i=1}^{t} K_{1, n_i}$ and $n_i \geq 2$. Then*

$$\chi'_L(H) = \begin{cases} n+1, & \textit{for } n_1 = n_2 = \cdots = n_t = n \textit{ and } t \leq n+1, \\ \infty, & \textit{otherwise.} \end{cases}$$

Proof. By Theorem 2.1, $\chi'_L(H) \geq q$, where $q = \max\{\chi_L(K_{1, n_i}) : i \in [1, t]\}$. Since $\chi_L(K_{1, n_i}) = n_i + 1$, $q = \max\{n_i + 1 | i \in [1, t]\}$. If there is j such that $n_j < q$, then $\chi'_L(H) = \infty$, otherwise (the case of $n_i = n$ for all i) the following coloring shows that $\chi'_L(H) = n + 1$.

Let $H = \bigcup_{i=1}^{t} K_{1, n}$ and $V(H) = \{x_i : i \in [1, t]\} \bigcup_{i=1}^{t} A_i$, where x_i is the root vertex and A_i is the set of the end vertices in K_{1, n_i}. Now, define a coloring $c : V(H) \to \{1, 2, ..., n + 1\}$ such that

$$c(x_i) = i,$$

$$c(A_i) = [1, n + 1] \setminus \{i\}.$$

Let $\Pi = \{C_1, C_2, ..., C_{n+1}\}$ be the partition of $V(H)$ induced by c, where C_i is set of all vertices receiving color i. Next, we will show that the color codes of all vertices are distinct. Let u and v be two distinct vertices with $c(u) = c(v)$. If $u = x_i$ and $v \in A_j$ for $i, j \in [1, t]$, then $d(u, C_{i+2 \bmod (n+1)}) = 1$ and $d(v, C_{i+2 \bmod (n+1)}) = 2$. Therefore, $c_\Pi(u) \neq c_\Pi(v)$. Now, assume $u \in A_i$ and $v \in A_j$, where $i \neq j$ and $i, j \in [1, t]$. Then $d(u, C_i) = 1$ and $d(v, C_i) = 2$. Therefore, $c_\Pi(u) \neq c_\Pi(v)$. Consequently, all vertices have distinct color codes, and so $\chi'_L(H) = n + 1$. □

Next, we will determine some necessary conditions for a disjoint union of graphs having finite locating-chromatic number.

Theorem 2.4. *Let* $H = \bigcup_{i=1}^m G_i$ *be a disconnected graph. If* $\chi'_L(H) < \infty$, *then* H *does not contain any two components* G_i *and* G_j *such that* $\chi_L(G_i) = |G_i|$, $\chi_L(G_j) = |G_j|$ *and* $|G_i| \neq |G_j|$.

Proof. For a contradiction, let G_i and G_j be any two components of H, for some $1 \leq i, j \leq m$ such that $\chi_L(G_i) = |G_i| = m$, $\chi_L(G_j) = |G_j| = n$, and $|G_i| \neq |G_j|$. Let $|G_i| < |G_j|$. We have $q = \max\{\chi_L(G_i) | i \in [1, m]\} \geq |G_j|$ and $r = \min\{|G_i| | i \in [1, m]\} \leq |G_i|$. So, $|G_j| \leq q \leq \chi'_L(H) \leq r \leq |G_i|$. It is a contradiction with $|G_i| < |G_j|$. Thus, H does not contain any two components G_i and G_j such that $\chi_L(G_i) = |G_i|$, $\chi_L(G_j) = |G_j|$ and $|G_i| \neq |G_j|$. □

Theorem 2.5. *Let* H *be a disconnected graph. If* $\chi'_L(H) < \infty$ *and* H *contains* K_n *as a component, then every other component must be not complete and each component has order at least* n.

Proof. Since $\chi'_L(H) < \infty$ and H contains a K_n as a component of H with $\chi_L(K_n) = n$, $\chi'_L(H) \geq n$. By Theorem 2.4, every other component must be not complete. Since $\chi'_L(H) < \infty$, every other component must have order at least n. □

Now, we will study the locating-chromatic number of a linear forest H, namely a disconnected graph with all components are paths.

Theorem 2.6. *Let* $H = \bigcup_{i=1}^{t} P_{n_i}$, $r = \min\{n_i \mid i \in [1, t]\}$. *If* $\chi'_L(H) < \infty$, *then* $3 \leq \chi'_L(H) \leq r$. *In particular*, $\chi'_L(H) = 3$ *is only satisfied by* $t = 1, 2$ *or* 3.

Proof. The first part is a direct consequence of Theorem 2.1. Now, let us prove the second part. Assume $\chi'_L(H) = 3$. Then $t \leq 3$. Since otherwise, there will be more than 3 dominant vertices, a contradiction. Let $V(H) = V(P_{n_1}) \cup V(P_{n_2}) \cup V(P_{n_3})$, where $V(P_{n_1}) = \{x_1, x_2, ..., x_{n_1}\}$, $V(P_{n_2}) = \{y_1, y_2, ..., y_{n_2}\}$ and $V(P_{n_3}) = \{z_1, z_2, ..., z_{n_3}\}$. Now, consider a coloring $c: V(H) \to \{1, 2, 3\}$ such that

$$c(x_1) = c(y_2) = c(z_1) = 1,$$

$$c(x_2) = c(y_1) = c(z_3) = 2,$$

$$c(x_3) = c(y_3) = c(z_2) = 3;$$

for $k \in [4, n_1]$, $l \in [4, n_2]$ and $m \in [4, n_3]$, define

$$c(x_k) = \begin{cases} 2, & \text{if } k \text{ is even,} \\ 3, & \text{if } k \text{ is odd;} \end{cases}$$

$$c(y_l) = \begin{cases} 1, & \text{if } l \text{ is even,} \\ 3, & \text{if } l \text{ is odd;} \end{cases}$$

$$c(z_m) = \begin{cases} 3, & \text{if } m \text{ is even,} \\ 2, & \text{if } m \text{ is odd.} \end{cases}$$

Let $\Pi = \{C_1, C_2, C_3\}$ be the partition induced by c. Next, we will show that color codes of all vertices are distinct (see Figure 1). It is clear that $c_\Pi(x_1) = (0, 1, 2)$, $c_\Pi(x_2) = (1, 0, 1)$, $c_\Pi(x_3) = (2, 1, 0)$, $c_\Pi(y_1) = (1, 0, 2)$, $c_\Pi(y_2) = (0, 1, 1)$, $c_\Pi(y_3) = (1, 2, 0)$, $c_\Pi(z_1) = (0, 2, 1)$, $c_\Pi(z_2) = (1, 1, 0)$, $c_\Pi(z_3) = (2, 0, 1)$. For $k \in [4, n_1]$, $c_\Pi(x_k) = (k-1, 0, 1)$ if k is even and $c_\Pi(x_k) = (k-1, 1, 0)$ if k is odd. For $l \in [4, n_2]$, $c_\Pi(y_l) = (0, l-1, 1)$ if l is even and $c_\Pi(y_l) = (1, l-1, 0)$ if l is odd. For $m \in [4, n_2]$, $c_\Pi(z_m) = (m-1, 1, 0)$ if m is even and $c_\Pi(z_m) = (m-1, 0, 1)$ if m is odd.

Therefore, all vertices have distinct color codes. Consequently, $\chi'_L(H) \leq 3$. If $t \geq 2$, then restrict the coloring c on the corresponding components. Thus, $\chi'_L(H) = 3$ for $t = 1, 2$ or 3.

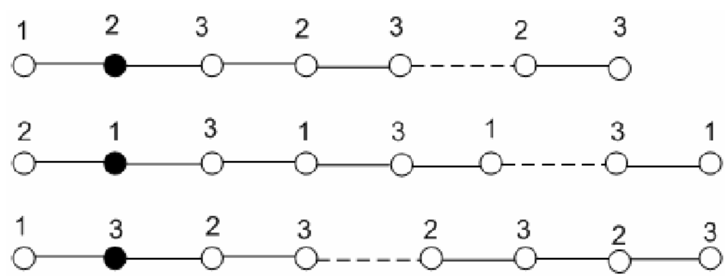

Figure 1. A locating-coloring of $H = P_{n_1} \cup P_{n_2} \cup P_{n_3}$.

□

Definition 2.7. Let G be a group and Ω be a generating set. The Cayley graph $\Gamma = \Gamma(G, \Omega)$ is the simple directed graph whose vertex-set and arc-set are defined as follows: $V(\Gamma) = G$; $E(\Gamma) = \{(g, gs) | s \in \Omega\}$.

Simple verifications show that $E\Gamma$ is well-defined. If $g^{-1} \in \Omega$ for every $g \in \Omega$, then $\Gamma(G, \Omega)$ is an undirected graph.

Theorem 2.8. *Let P_n be a path on n vertices. Let $H = kP_n$, $k \geq 1$ and $n \geq 4$. If $\chi'_L(H) \leq n$, then $k \leq \frac{n!}{2^m}$, where $m = \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Let $H = kP_n$, $k \geq 1$ and $n \geq 4$. To get the largest number *k*, we may assume that $\chi'_L(H) = n$. Let *c* be a locating *n*-coloring of *H*. Then the locating *n*-coloring *c* restricted to any component of *H* will be a permutation on $\{1, 2, ..., n\}$. Since *c* is a locating-coloring of *H*, the color codes for all vertices in *H* must be different. Therefore, the largest integer *k* is equal to the maximum number of *n*-permutations that can be used to color all the components of *H* such that the color codes of all vertices in *H* are different.

Let S_n be the symmetric group on $\{1, 2, ..., n\}$. Write an element $\sigma \in S_n$ as the permutation $(\sigma(1)\sigma(2)\cdots\sigma(n))$. Now, let G_n be a graph with vertex set $V(G_n) = S_n$, where two vertices $\sigma, \tau \in S_n$ are adjacent if and only if there exists $i \in [1, n]$ such that $|\sigma^{-1}(i) - \sigma^{-1}(j)| = |\tau^{-1}(i) - \tau^{-1}(j)|$ for all $j \in [1, n]$. Then the connected component X_n of G_n that contains identity permutation is given by

$$X_n = \begin{cases} \{\sigma \in S_n : (-1)^{\sigma(i)} = (-1)^i (1, 2, ..., n)\}, & \text{if } n \text{ odd}, \\ \{\sigma \in S_n : (-1)^{\sigma(i)} = (-1)^i (1, 2, ..., n)\} & \\ \bigcup \{\sigma \in S_n : (-1)^{\sigma(i)} = -(-1)^i (1, 2, ..., n)\}, & \text{if } n \text{ even}. \end{cases}$$

In particular,

$$|X_n| = \begin{cases} m!(m+1)!, & \text{if } n = 2m+1, \\ 2(m!)^2, & \text{if } n = 2m. \end{cases}$$

Let Γ_n be the induced subgraph of G_n on X_n. Note that X_n is a subgroup of S_n and Γ_n is a Cayley graph. Let $m = \left\lfloor \frac{n}{2} \right\rfloor$ and $l = \left\lceil \frac{n}{2} \right\rceil$. Then

$$Q_n = \{\sigma \in X_n : |\sigma^{-1}(l) - \sigma^{-1}(j)| = |l - j| (j = 1, 2, ..., n)\}$$

is a clique of size 2^m that contains the identity permutation. Thus, the independence number $\alpha(\Gamma_n)$ of Γ_n satisfies

$$\alpha(\Gamma_n) \leq \begin{cases} \frac{(m+1)!\,m!}{2^m} & \text{for } n = 2m+1, \\ \frac{(m!)^2}{2^{m-1}}, & \text{for } n = 2m. \end{cases}$$

At most one of any two adjacent permutations in Γ_n can be used to color one component of H such that all color codes in H are different. Therefore, the number k is bounded by the independence number $\alpha(\Gamma_n)$ times the number of components of G_n. This implies that $k \leq \alpha(\Gamma_n) \times \frac{n!}{|X_n|} = \frac{n!}{2^m}$, where $m = \left\lfloor \frac{n}{2} \right\rfloor$. □

In the following, we will determine the connected component X_n of G_n. We list the generating set Ω_n, the largest clique Q_n containing the identity permutation and the independent set A_n of $\Gamma_n(X_n, \Omega)$ for $n = 4$, 5, 6 and 7, as shown in Tables 1-4.

Table 1. The Ω_4, Q_4 and A_4 of $\Gamma_4(X_4, \Omega_4)$

$n = 4$	$\lvert X_4 \rvert = 8$
X_4	{(1234), (3214), (1432), (3412), (4321), (4123), (2341), (2143)}
Ω_4	{(3214), (1432), (2341), (4123), (4321)}
Q_4	{(1234), (3214), (4321), (4123)}
A_4	{(1234), (3412)}

Table 2. The Ω_5, Q_5 and A_5 of $\Gamma_5(X_5, \Omega_5)$

$n = 5$	$\mid X_5 \mid = 12$
X_5	{(12345), (32145), (14325), (12543), (54321), (34125), (14523), (52341), (34521), (54123), (32541), (52143)}
Ω_5	{(32145), (14325), (12543), (52341), (34521), (54123), (54321)}
Q_5	{(12345), (14325), (52341), (54321)}
A_5	{(12345), (34125), (14523)}

Next, the Cayley graphs Γ_n for $n = 4, 5$ can be seen in Figures 2 and 3.

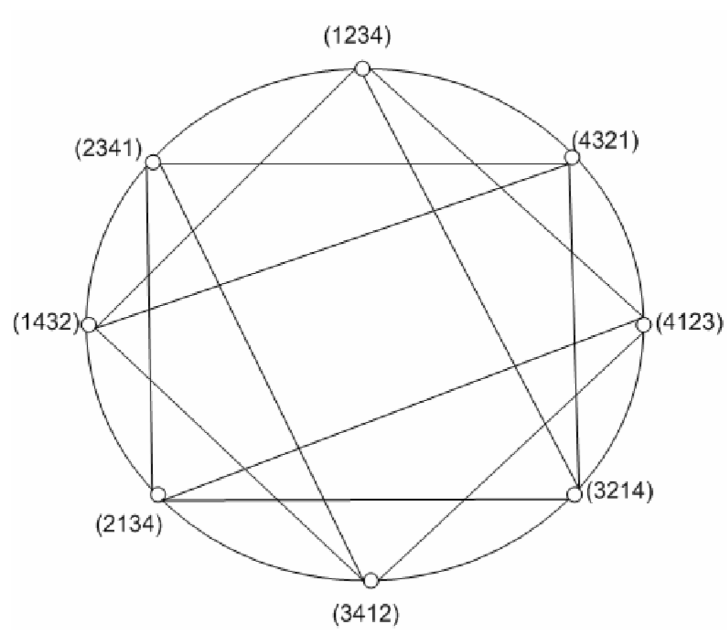

Figure 2. The Cayley graph Γ_4.

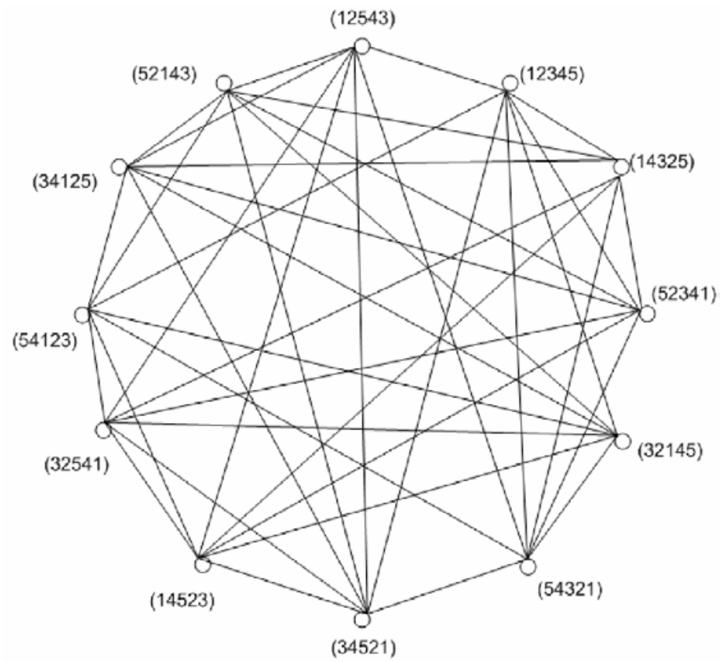

Figure 3. The Cayley graph Γ_5.

For $n = 6$ and 7, Tables 3 and 4 give the generating set Ω_n, the largest clique Q_n and the largest independent set A_n of $\Gamma_n(X_n, \Omega_n)$.

Table 3. The Ω_6, Q_6 and A_6 of $\Gamma_6(X_6, \Omega_6)$

$n = 6$	$\mid X_6 \mid = 72$
Ω_6	{(321456), (143256), (125436), (123654), (523416), (163452), (612345), (234561), (654123), (456321), (652341), (634521), (654321)}
Q_6	{(123456), (143256), (523416), (543216), (654321), (652341), (614325), (612345)}
A_6	{(123456), (143652), (163254), (325416), (345612), (365214), (521436), (541632), (561234)}

Table 4. The Ω_7, Q_7 and A_7 of $\Gamma_7(X_7, \Omega_7)$

$n = 7$	$\lvert X_7 \rvert = 144$
Ω_7	{(3214567), (1432567), (1254367), (1236547), (1234765), (5234167), (1634527), (7234561), (1274563), (3456721), (7612345), (5674321), (7654123), (7652341), (7456321), (7654321)}
Q_7	{(1234567), (1654327), (1254367), (7254361), (7234561), (7634521), (1634527), (7654321)}
A_7	{(1234567), (1436527), (1632547), (3254167), (3456127), (3652147), (5234761), (5436721), (5632741), (1254763), (1456723), (1652743), (5214367), (5416327), (5612347), (3214765), (3416725), (3612745)}

Theorem 2.9. *Let* P_4 *be a path on* 4 *vertices. If* $H = kP_4$, *then*

$$\chi'_L(H) = \begin{cases} 3, & \text{for } 1 \le k \le 3, \\ 4, & \text{for } 4 \le k \le 6, \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.6, $\chi'_L(H) = 3$, for $1 \le k \le 3$ and $\chi'_L(H) \ge 4$, for $k \ge 4$. By Theorem 2.8, $\chi'_L(H) \le 4$, for $n \le 6$. Therefore, $\chi'_L(H) = 4$, for $4 \le k \le 6$ and $\chi'_L(H) = \infty$ if $k \ge 7$. The locating-coloring of kP_4, for $k = 4, 5$ or 6, can be taken from Figure 4:

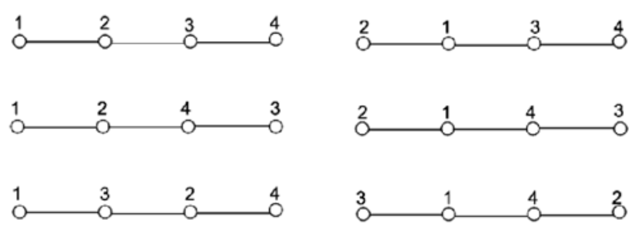

Figure 4. The locating 4-coloring of $H = 6P_4$.

□

Theorem 2.10. *Let* P_5 *be a path on* 5 *vertices. If* $H = kP_5$, *then*

$$\chi'_L(H) = \begin{cases} 3, & \text{for } 1 \le k \le 3, \\ 4, & \text{for } 4 \le k \le 7, \\ 4 \text{ or } 5, & \text{for } 8 \le k \le 30, \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.6, $\chi'_L(H) = 3$, for $1 \le k \le 3$ and $\chi'_L(H) \ge 4$, for $k \ge 4$. Since we can have the locating 4-coloring on kP_5 for $k = 4, 5, 6$ or 7 as shown in Figure 5, $\chi'_L(H) = 4$, for $4 \le k \le 7$. By Theorem 2.8, if $\chi'_L(H) \le 5$, then $n \le 30$. Therefore, we have $\chi'_L(H) = 4$ or 5, for $8 \le k \le 30$.

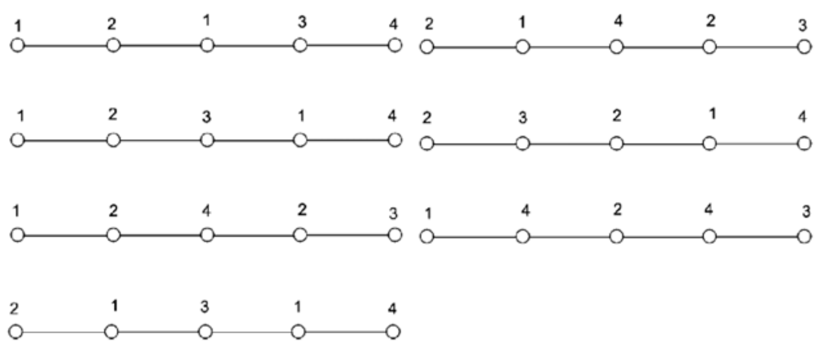

Figure 5. The locating 4-coloring of $H = 7P_5$.

□

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